On sheaves on semicartesian quantales and their truth values

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Abstract

In this paper, we introduce a new definition of sheaves on semicartesian quantales, providing first examples and categorical properties. We note that our sheaves are similar to the standard definition of a sheaf on a locale; however, we prove that in general it is not an elementary topos—since the lattice of external truth values of $Sh(Q)$, $Sub(1)$, is canonically isomorphic to the quantale Q—placing this paper as part of a greater project towards a monoidal (not necessarily cartesian) closed version of elementary topos. To start the study the logical aspects of the category of sheaves we are introducing, we explore the nature of the 'internal truth value objects' in such sheaves categories. More precisely, we analyse two candidates for subobject classifier for different subclasses of commutative and semicartesian quantales.

Keywords: Quantales, sheaves, external truth values, internal truth values

1 Introduction

Sheaf Theory is a well-established area of research with applications in Algebraic Topology [10], Algebraic Geometry [12], Geometry [14], Logic [16] and others. A sheaf on a locale L is a functor $F : L^{op} \to Set$ that satisfies certain gluing properties expressed by an equalizer diagram. However, for quantales—a non-idempotent and non-commutative generalization of locales introduced by C.J. Mulvey [18]—there are many definitions of sheaves on quantales: in [8], sheaves on quantales are defined with the goal of forming Grothendieck toposes from quantales. In [17], the sheaf definition preserves an intimate relation with Q-sets, an object introduced in the paper as a proposal to generalize Ω-sets, defined in [11], for Ω a complete Heyting algebra.[1] More recently, in [2], sheaves are functors that make a certain diagram an equalizer. Besides, an extensive work about sheaves on *involutive quantale*, which goes back to ideas of Bob Walters [24], was recently studied by Hans Heymans, Isar Stubbe [13] and Pedro Resende [21], for instance.

[1] Given a proper notion of morphisms of Ω-sets, the category of Ω-sets is equivalent to the category of sheaves on Ω.

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In this paper, we provide the basic definitions and properties of quantales in Section 2, introducing the notions of Artinian and geometric quantales with examples that use von Neumann regular rings and the extended half-line $[0, +\infty]$, respectively. In Section 3, we provide a new definition of sheaves on semicartesian quantales (as a functor that forms an equalizer diagram), given first examples and categorical properties. We note that our sheaves are similar to the standard definition of a sheaf; however, we prove in Section 4, that in general it is not an elementary topos—since the lattice of external truth values of $Sh(Q)$, $Sub(1)$, is canonically isomorphic to the quantale Q—placing this paper as part of a greater project towards a monoidal closed but non-cartesian closed version of elementary topos. To explore our interest in the logical aspects of the category of sheaves we are introducing, we provide in Section 5 a detailed construction and analysis of two candidates for subobject classifier for different subclasses of commutative and semicartesian quantales: in other words, we investigate the nature of the 'internal truth value objects' in such sheaves categories.

2 Quantales

DEFINITION 1
A **quantale** is a structure $Q = (Q, \leq, \odot)$ where: $(Q, \leq)$ is a complete lattice; $(Q, \odot)$ is a semigroup—i.e. the binary operation $\odot : Q \times Q \to Q$ (called multiplication) is associative; moreover, $Q = (Q, \leq, \odot)$ satisfies the following distributive laws: for all $a \in Q$ and $\{b_i\}_{i \in I} \subseteq Q$

$$a \odot (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \odot b_i) \quad \text{and} \quad (\bigvee_{i \in I} b_i) \odot a = \bigvee_{i \in I} (b_i \odot a)$$

REMARK 1
Note that:

1. In any quantale Q the multiplication is increasing in both entries, where increasing in the second entry means that given $a, b, b' \in Q$ such that $b \leq b'$, we have $a \odot b \leq a \odot b'$. Indeed, $a \odot b' = a \odot (b \vee b') = (a \odot b) \vee (a \odot b')$, thus $a \odot b \leq a \odot b'$.
2. Since the least element of the quantale Q, here denoted by 0 (or $\perp$), is also the supremum of the emptyset, note that $a \odot 0 = 0 = 0 \odot a, \forall a \in Q$.

Similarly, a *unital* quantale is a structure $(Q, \leq, \odot, 1)$, where $(Q, \odot, 1)$ is a *monoid*. Note that the associativity of $\odot$ and the identity element provide a (strict) monoidal structure to $(Q, \leq, \odot, 1)$, viewed as a poset category.

Let $(Q, \leq)$ be a quantale where the multiplication is the infimum ($\odot = \wedge$). Then $\odot$ is a commutative operation where $\top$, the largest member of Q, is its identity element. In such a case, we obtain a **locale**. Thus, every locale is a unital quantale. The main example of locale used in sheaf theory is the locale $\mathcal{O}(X)$ of open subsets of a topological space X, where the order relation is given by the inclusion, the supremum is the union and the finitary infimum is the intersection. We list below some examples of unital quantales that are not locales:

EXAMPLE 1 (Quantales)

1. The extended half-line $[0, \infty]$ with order $\geq$, and the usual sum of real numbers as the multiplication. Since the order relation is $\geq$, the top element is 0 and the bottom elements is ∞;

2. The extended natural numbers $\mathbb{N} \cup \{\infty\}$, with the same quantalic structure of $[0, \infty]$;
3. The set $\mathcal{I}(R)$ of ideals of a commutative and unital ring R with order $\subseteq$, the inclusion of ideals and the multiplication as the multiplication of ideals. The supremum is the sum of ideals, the top element is R and the trivial ideal is the bottom;
4. The set of closed right (or left) ideals of a unital C^*-algebra, the order is the inclusion of closed right (or left) ideals and the multiplication is the topological closure of the multiplication of the ideals.

For more details and examples, we recommend [22].

DEFINITION 2
A quantale $Q = (Q, \leq, \odot)$ is

1. **commutative** when $(Q, \odot)$ is a commutative semigroup;
2. **idempotent** when $a \odot a = a$, for $a \in Q$;
3. **right-sided** when $a \odot \top = a$, for all $a \in Q$, where $\top$ is the top member of the poset;
4. **semicartesian** when $a \odot b \leq a, b$, for all $a, b \in Q$;
5. **integral** when Q is unital and $1 = \top$;
6. **divisible** when $a \leq b \implies \exists x, b \odot x = a,$;
7. **strict linear** when $\leq$ is a linear order and $a \neq 0, b \neq c \implies a \odot b \neq a \odot c$, for all $a, b, c \in Q$.

The quantales $[0, \infty]$, $\mathbb{N} \cup \{\infty\}$ and $\mathcal{I}(R)$ are commutative and integral unital quantales. Besides, they are also divisible (for $\mathcal{I}(R)$, we have to take R a PID). The last example is neither commutative nor semicartesian but it is right-sided (resp. left-sided) quantale. [22].

Notation: We denote the set of all idempotent elements of a quantale Q by $Idem(Q) = \{a \in Q : a \odot a = a\}$.

REMARK 2
Note that:

1. A quantale $(Q, \leq, \odot)$ is semicartesian iff $a \odot b \leq a \wedge b$, for all $a, b \in Q$.
2. Let Q be a unital quantale, then it is integral iff it is semicartesian. Indeed: suppose that Q is integral, since $b \leq \top$ we have $a \odot b \leq a \odot \top = a \odot 1 = a$, then Q is semicartesian; conversely, suppose that Q is semicartesian, since $\top = \top \odot 1 \leq 1$, then $\top = 1$.
3. Let Q be an integral commutative quantale. Define a binary relation on Q by: $x \preceq y \Leftrightarrow x = x \odot y$. Then: (i) $\preceq$ is a partial order; (ii) $x \preceq y \Rightarrow x \leq y$; (iii) $x \preceq y, y \leq z \Rightarrow x \preceq z$; (iv) $e \in Idem(Q) \Rightarrow (e \preceq x \Leftrightarrow e \leq x)$.

The following result explains why we claim that our quantale are orthogonal to idempotent quantales. Therefore, the definition of sheaves on quantales that we will introduce in Section 3 is orthogonal to notions of sheaves on idempotent quantales.

PROPOSITION 1
If Q is a semicartesian quantale, then by adding the idempotency we obtain that $\odot = \wedge$. In other words, Q is a locale.

PROOF. Since Q is semicartesian, by Remark 2, for any b, c in Q, $b \odot c \leq b, c$. If Q is idempotent, for any $a \in Q$ such that $a \leq b$ and $a \leq c$ we have $a = a \odot a \leq b \odot c$, because the multiplication is increasing in both entries. So if Q is semicartesian and idempotent, by the transitivity of the order relation:

$$a \leq b \odot c \iff a \leq b \text{ and } a \leq c.$$

Thus, the multiplication satisfies the definition of the meet operation. □

The above proposition is just a particular case of [19, Proposition 2.1]. Observe that any notion of sheaf on a idempotent and semicartesian quantale is necessarily a sheaf on a locale, which is a well-established object of study.

Construction of quantales:

1. Notice that given a family of quantales $\{Q_i : i \in I\}$, the cartesian product $\prod_{i \in I} Q_i$ with component-wise order (i.e. $(a_i)_i \leq (b_i)_i \iff a_i \leq b_i, \forall i \in I$) is a quantale. Define $\bigvee_{j \in I}(a_{ij})_i = (\bigvee_{j \in I} a_{ij})_i$, $\bigwedge_{j \in I}(a_{ij})_i = (\bigwedge_{j \in I} a_{ij})_i$ and $(a_i)_i \odot (b_i)_i = (a_i \odot b_i)_i$. All verifications are straightforward, but we will check one of the distributive laws:
$$\bigvee_{j \in I}(a_i \odot b_{ij})_i = (\bigvee_{j \in I} a_i \odot b_{ij})_i = (a_i \odot \bigvee_{j \in I} b_{ij})_i = (a_i)_i \odot \bigvee_{j \in I}(b_{ij})_i.$$
It is easy to see that $\prod_{i \in I} Q_i$ is a semicartesian/commutative quantale whenever each Q_i is a semicartesian/commutative quantale.
2. If Q is a commutative semicartesian quantale, it is straightforward to check that given $l \in Idem(Q)$ and $u \in Q$ such that $l \leq u$, then the subset $[l, u] = \{x \in Q : l \leq x \leq u\}$ is closed under $\bigvee$ and $\odot$, thus it determines a 'interval subquantale' that is also semicartesian and commutative.

An example of a semicartesian quantale that is not integral is constructed as follow: let Q be a integral and not idempotent quantale. Given $a \in Q \setminus Idem(Q)$, then the interval $[\bot, a]$ is a non unital semicartesian quantale.

Remark 3

Every commutative unital quantale Q can be associated to a closed monoidal symmetric[2] poset category Q, where there exists a unique arrow in $Hom(a, b)$ iff $a \leq b$. Note that the product $\prod$, coproduct $\coprod$ and tensor $\otimes$ are defined, respectively, by the infimum, $\bigwedge$, the supremum $\bigvee$ and the dot $\odot$ and the 'exponential' is given by $b^a = \bigvee\{c \in Q : a \odot c \leq b\}$, where b^a is an alternative notation for $a \to b$.

Now, we introduce an operation that sends elements of a commutative and semicartesian quantale Q into idempotent elements in the locale $Idem(Q)$. In those conditions, we define

$$q^- := \bigvee\{p \in Idem(Q) : p \leq q \odot p\}.$$

Since Q is semicartesian and commutative, note that $p \leq q \odot p$ iff $p = q \odot p = p \odot q$.

Now, we list properties of $(-)^- : Q \to Idem(Q)$.

[2] A monoidal category is category equipped with a tensor functor and satisfies coherence laws given by commutative diagrams; the symmetry establishes a kind of commutativity for the tensor; and closed means that there is an isomorphism $Hom(a \otimes c, b) \cong Hom(c, b^a)$, and under such bijection the arrow $ev \in Hom(a \otimes b^a, b)$ that corresponds to $id \in Hom(b^a, b^a)$.

PROPOSITION 2
If Q is a commutative and semicartesian quantale, and $\{q_i : i \in I\} \subseteq Q$, then

(1) $0^- = 0$ and $1^- = 1$ (if Q is unital)
(2) $q^- \leq q$
(3) $q^- \odot q = q^-$
(4) $q = q^- \Leftrightarrow q \odot q = q$
(5) $q^- \odot q^- = q^-$
(6) $q^- = max\{e \in Idem(Q) : e \leq q\}$
(7) $q^{-^-} = q^-$
(8) $p \leq q$ and $x \odot p = x$, then $x \odot q = x$
(9) $p \leq q \Rightarrow p^- \leq q^- \Leftrightarrow p^- \odot q^- = p^-$
(10) $(a \odot b)^- = a^- \odot b^-$
(11) $q_j{}^- \odot \bigvee_{i \in I} q_i = q_j{}^-$
(12) $\bigvee_i q_i^- \leq (\bigvee_i q_i)^-$

PROOF.

1. Straightforward.
2. If $e \in Idem(Q)$ is such that $e \leq q \odot e$, then $e \leq q \odot e \leq q$, since Q is semicartesian. Thus, by the definition of q^- as a least upper bound, $q^- \leq q$.
3. Since multiplication distributes over arbitrary joins,
$$q^- \odot q = \bigvee\{p \odot q : p = q \odot p, p \in Idem(Q)\} = \bigvee\{p \in Idem(Q) : p = q \odot p\} = q^-$$
4. ($\Rightarrow$) From the previous item.
($\Leftarrow$) By maximality, $q \leq q^-$. Thus, the result follows from item (2).
5. Since multiplication distributes over arbitrary joins,
$$\begin{aligned} q^- \geq q^- \odot q^- &= \bigvee\{p \odot q^- Idem(Q) : p \in Idem(Q), p = q \odot p\} \\ &= \bigvee\{p \odot p' : p, p' \in Idem(Q), p = q \odot p, p' = q \odot p'\} \\ &\geq \bigvee\{p \odot p : p \in Idem(Q), p = q \odot p\} \\ &= \bigvee\{p : p \in Idem(Q), p = q \odot p\} \\ &= q^- \end{aligned}$$
6. By items (2) and (5), $q^- \in \{e \in Idem(Q) : e \leq q\}$. If $e \in Idem(Q)$ is such that $e \leq q$, then $e = e \odot e \leq q \odot e \leq e$, thus $e = e \odot q$; then $e \leq q^-$, by the definition of q^- as a l.u.b.
7. By item (2), $q^{-^-} \leq q^-$. On the other hand, by items (4) and (5) and maximality of q^{-^-}, we have $q^- \leq q^{-^-}$.
8. Since $x = x \odot p \leq x \odot q \leq x$.
9. Suppose $p \leq q$. Then by items (3) and (8), $p^- \odot q = p^-$.
By item (5), $p^- \in Idem(Q)$ and, by maximality of q^-, $p^- \leq q^-$.
Since $p^-, q^- \in Idem(Q)$ (item (5)) then, by the argument in the proof of item (6), we have $p^- \leq q^-$ iff $p^- \odot q^- = p^-$.

10. Note that $a^- \odot b^-$ is an idempotent such that $a^- \odot b^- \odot a \odot b = a^- \odot b^-$. So $(a \odot b)^- \geq a^- \odot b^-$. On the other hand, by item (9), $(a \odot b)^- \odot a^- = (a \odot b)^- = (a \odot b)^- \odot b^-$. Then, $(a \odot b)^- \odot (a^- \odot b^-) = ((a \odot b)^- \odot a^-) \odot b^- = (a \odot b)^- \odot b^- = (a \odot b)^-$. Thus, $(a \odot b)^- \leq a^- \odot b^-$.
11. Since $q_j^- = q_j^- \odot q_j \leq q_j^- \odot \bigvee_{i \in I} q_i \leq q_j^-$.
12. Since $q_j \leq \bigvee_i q_i$, from the item (9) we obtain $q_j^- \leq (\bigvee_i q_i)^-$, and then $\bigvee_j q_j^- \leq (\bigvee_i q_i)^-$, by sup definition. □

PROPOSITION 3
Let Q be a commutative and integral (unital) quantale. Consider the maps $i : Idem(Q) \hookrightarrow Q$ and $(\ )^- : Q \to Idem(Q)$, then:

1. $(Idem(Q), \bigvee, \odot, 1)$ is a locale and the inclusion map $i : Idem(Q) \hookrightarrow Q$ preserves $\odot$, sups and $\top$.
2. The map $(\ )^- : Q \to Idem(Q)$ preserves $\odot$ and $\top$.
3. The adjunction relations (for posets) holds for each $e \in idem(Q)$ and $q \in Q$

$$Hom_Q(i(e), q) \cong Hom_{idem(Q)}(e, q^-)$$

PROOF.

1. The sup of a set of idempotents is an idempotent (in the same vein of the proof of item (5) in the previous proposition). If $f, e, e' \in Idem(Q)$, then $e \odot e' \leq e, e'$, since Q is semicartesian. Moreover, if $f \leq e, e'$, then $f = f \odot f \leq e \odot e'$. Thus, $e \odot e'$ is the g.l.b. of e, e' in $Idem(Q)$. The other claims are straightforward.
2. This is contained in items (1) and (10) of the previous proposition.
3. Since we are dealing with posets, it is enough to show that, for each $q \in Q, e \in Idem(Q)$,

$$i(e) \leq q \iff e \leq q^-.$$

If $i(e) \leq q$, then by item (6) in the previous proposition $e \leq q^-$.
On the other hand, if $e \leq q^-$, then by item (4) and the equivalence in the (9) in the previous proposition $e = e \odot q^-$. Then, by item (2), $e \leq e \odot q \leq q$. □

DEFINITION 3
Let *Loc* be the category of locales with morphisms that preserve finitary infs and arbitrary sups, and *CSQ* the category of commutative semicartesian quantales with morphisms that preserve sups and $\top$ satisfying that $f(a \odot b) \geq f(a) \odot f(b)$.

The next proposition is analogous to the previous one, but for the category *CSQ* instead of the poset category of commutative and semicartesian quantales.

PROPOSITION 4
Consider inclusion functor $\iota : Loc \hookrightarrow CSQ$. Then:

1. The inclusion functor $\iota : Loc \hookrightarrow CSQ$ is full and faithful.
2. $Q \mapsto Idem(Q)$ determines the right adjoint of the inclusion functor $\iota : Loc \hookrightarrow CSQ$, where the inclusion $i_Q : \iota(Idem(Q)) \hookrightarrow Q$ is a component of the co-unity of the adjunction.

Observe that the last item follows the same idea of Lemma 2.2 in [19].

DEFINITION 4
A set o elements $\{q_i : i \in I\}$ of Q is a **partition of** $q \in Q$ if $\bigvee_{i\in I} q_i = q$ and $q_i \odot q_j = 0$, for each $i \neq j$.

It is clear that if $\{q_i : i \in I\}$ is a partition of q, then $\{q_i \odot a : i \in I\}$ is a partition of $a \odot q$, for any $a \in Q$. Thus, every partition of unity determines a partition for any $q \in Q$.

EXAMPLE 2
For a commutative ring A, any ideal I has a partition: take an idempotent $e \in A$ and observe that $1 = e+1-e$. We have $(1) = A$ (the unity), $(e+1-e) = (e)+(1-e)$ and $(e)\odot(1-e) = (e)\cap(1-e) = 0$. So $\{(e), (1-e)\}$ is a partition of (1) and from it we obtain a partition $\{(e) \odot I, (1-e) \odot I\}$ for any ideal I. If A only have trivial idempotents then the ideals admit trivial partition.

Next we explore other properties of the construction $q \mapsto q^-$ in a more specific class of quantales.

DEFINITION 5
We say that a (commutative, semicartesian) quantale Q is:

1. An **Artinian quantale** if each infinite descending chain $q_0 \geq q_1 \geq q_2 \geq q_3 \geq \dots$ stabilizes for some natural number $n \in \mathbb{N}$, which may vary according to the chain.
2. A **p-Artinian quantale** if for each $q \in Q$, the infinite descending chain of powers of q, $q^1 \geq q^2 \geq q^3 \geq \dots$, stabilizes for some natural number $n \in \mathbb{N} \setminus \{0\}$, which may vary according to the chain.
3. If there is a natural number $n \geq 1$ such that for all $q \in Q$ we have $q^n = q^{n+1}$, then we say that Q is **uniformly p-Artinian**. The least $n \in \mathbb{N}$ such that, for each $q \in Q$, $q^{n+1} = q^n$ is called the **degree of Q**.

The following results are straightforward.

REMARK 4
Let Q be a commutative and semicartesian quantale.

1. If Q is Artinian or uniformly p-Artinian, then Q is p-Artinian.
2. If Q is a p-Artinian quantale, $q \in Q$ and $q^{n+1} = q^n$, then $q^- = q^n$.

The example that motivates such terminology is the set of ideals of an Artinian commutative unitary ring. Concerning this example, we add the following:

PROPOSITION 5
Let A be a commutative unitary ring and consider $Q = \mathcal{I}(A)$ be its quantale of all ideals. Consider:

1. $\mathcal{I}(A)$ is p-Artinian;
2. For each $a \in A$, there is $n \in \mathbb{N}$ such that $(a)^n = (a)^{n+1}$;
3. For each $a \in A$, there is $n \in \mathbb{N}$ and $b \in A$ that $a^n = b.a^{n+1}$. This means that A is strongly π-regular.
4. Each prime proper ideal of A is maximal;
5. $A/nil(A)$ is a von Neumann regular ring;

   Then we have the following implications:

$$1 \implies 2 \iff 3 \implies 4 \iff 5.$$

Moreover, if A is a reduced ring (i.e. $nil(A) = \{0\}$), then all items above are equivalent between them and also are equivalent to

6. $\mathcal{I}(A)$ is a uniformly p-Artinian quantale of degree 2.

PROOF. 1 $\implies$ 2 : By the definition of p-Artinian.

2 $\iff$ 3 : Straightforward.

3 $\implies$ 4 : Let P be a prime proper ideal and take $a \notin P$. By 3, there is $n \in \mathbf{N}$ and $b \in A$ such that $a^n - ba^{n+1} = 0$. So $a^n(1-ba) = 0$. Since $a^n \notin P$ and P is prime, we have that $a^n(1-ba) \in P$. Then $1 \in P + Ra$ and we obtain $A = P + Aa$. In other words, every non-zero element in A/P is invertible, which means that A/P is field and therefore P is maximal.

4 $\iff$ 5 This is stated in [15, Exercise 4.15], where Krull dimension 0 means precisely that all prime ideals are maximal ideals.

Now, suppose that $\mathcal{I}(A)$ is uniformly p-Artinian. This gives that $\mathcal{I}(A)$ is p-Artinian and so we do not have verify 1. We conclude the sequence of implications by showing that 5 implies 2: we use that a ring is von Neumann regular iff every principal left ideal is generated by an idempotent element. Since A is commutative and $A/nil(A) = A$ is von Neumann regular, for each $a \in A = A$, $(a) = (e)$ for some idempotent $e \in A$. Therefore, $(a)^n = (e)^n = (e)^{n+1} = (a)^{n+1}$. □

EXAMPLE 3
Since A is a strongly (von Neumann) regular ring if and only if A is a reduced regular ring [20, Remark 2.13], any reduced regular ring satisfies condition 3 (every regular ring is π-regular) so $\mathcal{I}(A)$ is an example of a uniformly p-Artinian quantale of degree 2.

REMARK 5
For commutative rings, strongly von Neumann regular is equivalent to von Neumann regular.

PROPOSITION 6
If Q is a uniformly p-Artinian quantale, and $(I, \leq)$ is an upward directed poset, then $(\bigvee_{i\in I} q_i^-) = (\bigvee_{i\in I} q_i)^-$.

PROOF. The relation $(\bigvee_{i\in I} q_i^-) \leq (\bigvee_{i\in I} q_i)^-$ holds in general.

Now suppose that the degree of Q is $n \in \mathbb{N}$. Then

$$(\bigvee_{i\in I} q_i)^- = (\bigvee_{i\in I} q_i)^n = \bigvee_{i_1,\cdots i_n \in I} q_{i_1} \odot \cdots \odot q_{i_n}.$$

But, since $(\bigvee_{i\in I} q_i)$ is an upward directed sup, for each $i_1, \cdots, i_n \in I$, there is $j \in I$ such that $q_{i_1}, \cdots, q_{i_n} \leq q_j$ then

$$\bigvee_{i_1,\cdots,i_n \in I} q_{i_1} \odot \cdots \odot q_{i_n} \leq \bigvee_{j\in I} q_j^n = (\bigvee_{j\in I} q_j^-).$$

□

Now, observe that the equality $\bigvee_{i\in I} q_i^- = (\bigvee_{i\in I} q_i)^-$ holds, in general (i.e. for each sup) for any locale, but not for any quantale.

EXAMPLE 4
If $Q = \mathbb{R}_+ \cup \{\infty\}$ is the extended half-line presented in 1, then all elements of Q are in the interval $[0, \infty]$. There are only two idempotent elements in this quantales, 0 and ∞. Since, in this case, the

supremum is the infimum, and $0 \geq 0 + q$ if and only if $q = 0$, we have

$$q^- = \begin{cases} 0, & \text{if } q = 0, \\ \infty, & \text{if } q \in (0, \infty]. \end{cases}$$

So, for a subset $\{q_i : q_i \neq 0, \forall i \in I\} \subseteq Q$ then $\bigvee_{i \in I}(q_i^-) = \infty$ but $(\bigvee_{i \in I} q_i)^-$ may be zero or ∞ depending if the supremum (which is the infimum in the usual ordering) of q_i's is zero or not.

Since some but not all quantales satisfies such equality, we provide a name for it.

DEFINITION 6
Let Q be a commutative semicartesian quantale, we say Q is a **geometric quantale** whenever $\bigvee_{i \in I} q_i^- = (\bigvee_{i \in I} q_i)^-$, for each $\{q_i : i \in I\} \subseteq Q$.

Locales satisfy, trivially, this geometric condition. Moreover:

EXAMPLE 5
The extended natural numbers presented in Example 1 is a geometric quantale. We argue in Example 4 that the extended positive real numbers is not a geometric quantale because $(\bigvee_{i \in I} q_i)^-$ could be zero. However, we only have $(\bigvee_{i \in I} q_i)^- = \infty$ since we are considering the subset $\{q_i : q_i \neq 0, \forall i \in I\} \subseteq \mathbb{N} \cup \{\infty\}$.

Note that the poset of all ideals of a PID *is not* a geometric quantale. In particular, $(\mathbb{N}, \cdot, \sqsubseteq)$, where $a \sqsubseteq b$ iff $b \mid a$, is not a geometric quantale.

We choose such terminology to indicate that, under those conditions on Q, the function $(-)^- : Q \to Idem(Q)$ is a *strong geometric morphism* of unital quantales, i.e. it preserves 1, $\odot$ and arbitrary sups. This coincides with the notion of a quantale (homo)morphism as defined e.g. in [22]. In [23, Section 3.4], different notions of morphisms of quantales are studied and applied; the results therein will be published in future work.

Moreover, we may construct geometric quantales from others geometric quantales:

PROPOSITION 7
The subclass of geometric quantales is closed under arbitrary products and interval construction.

PROOF. Given a family of quantales $Q = \{Q_i : i \in I\}$, the cartesian product $\prod_{i \in I} Q_i$ with component-wise order is a geometric quantale.

It follows from the fact that $(-)^-$ is component wise. Indeed,

$$\begin{aligned}
(q_i)_i^- &= \bigvee \{(p_i)_i \in Idem(Q) \,:\, (p_i)_i \leq (p_i)_i \odot (q_i)_i\} \\
&= \bigvee \{(p_i)_i \in Idem(Q) \,:\, p_i \leq p_i \odot q_i, \forall i \in I\} \\
&= (\bigvee \{(p_i \in Idem(Q) \,:\, p_i \leq p_i \odot q_i, \forall i \in I\})_i \\
&= (q_i^-)_i.
\end{aligned}$$

Then

$$\bigvee_{j \in I} (q_{ij})_i^- = \bigvee_{j \in I} (q_{ij}^-)_i = (\bigvee_{j \in I} (q_{ij}^-)_i = ((\bigvee_{j \in I} q_{ij})^-)_i. \qquad \square$$

We defined $(-)^-$ as a supremum and verified it is the best lower idempotent approximation, in the sense that q^- is the maximum of idempotents e such that $e \leq q$ (Proposition 2). Analogously, we are tempted to define an operation $(-)^+$ as an infimum and obtain that q^+ is the minimum of idempotents e such that $q \leq e$ (or, possibly, $q \preceq e$). To achieve this, we need **double-distributive quantales**, which are quantales that satisfy the following additional distributive law, for $I \neq \emptyset$:

$$a \odot (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \odot b_i).$$

Examples of double-distributive quantales are: locales; the extended half-line $[0, \infty]$ and the extended natural numbers $\mathbb{N} \cup \{\infty\}$; a subclass of the quantales of ideals of a commutative and unital ring that is closed under quotients and finite products and contains the principal ideal domains. Besides, double-distributive quantales are closed under arbitrary products and interval construction.

For members $u \in Q$ these quantales Q, there is a more explicit construction of u^- as a transfinite power u^α, where α is an ordinal with cardinality $\leq$ cardinality of Q.

PROPOSITION 8

Let Q be a unital double distributive commutative and semicartesian quantale. Given $q \in Q$, consider the transfinite chain of powers $(q^\alpha)_{\alpha \geq 1\ ordinal}$: $q^1 := q$; $q^{\alpha+1} := q^\alpha \odot q$; if $\gamma \neq 0$ is a limit ordinal, then $q^\gamma := \bigwedge_{\beta < \gamma} q^\beta$

1. It is a descending chain;
2. It stabilizes for some ordinal α, $q^\beta = q^\alpha$ for each $\beta \geq \alpha$, and $\alpha <$ successor($card(Q)$);
3. Moreover, if $q^\alpha = q^{\alpha+1}$, then $q^- = q^\alpha$.

PROOF.

1. This follows directly by induction.
2. Suppose that the restriction of the descending chain to all ordinal γ with $1 \leq \delta \leq \alpha$ is a strictly descending chain in Q. Thus, we have an injective function $[1, \alpha] \to Q$, $\delta \mapsto q^\delta$. Since $card(\alpha) = card([1, \alpha])$, we must have $card(\alpha) \leq card(Q)$, then $\alpha <$ successor($card(Q)$). Thus, in particular, there is a largest ordinal α such that $(q^\delta)_{1 \leq \delta \leq \alpha}$ is a strictly descending chain. Thus, $q^{\alpha+1} = q^\alpha$ and, by induction, $q^\beta = q^\alpha$ for each $\beta \geq \alpha$.
3. Suppose that the transfinite descending chain stabilizes at α (i.e. $q^{\alpha+1} = q^\alpha$). So $q^\alpha = q^\alpha \odot q^\alpha$ and $q^\alpha = q^{\alpha+1} = q^\alpha \odot q$ and $q^\alpha = q^\alpha \odot q^\alpha$. Thus, q^α is an idempotent such that $q^\alpha \preceq q$ (in particular, $q^\alpha \leq q^-$). On the other hand, for any idempotent $p \in Q$ such that $p \preceq q$ (i.e. $p = p \odot q$), we have, by induction $p \preceq q^\beta$, for all ordinal $\beta \geq 1$: in the induction step for ordinal limits, we have to use the hypothesis that Q is double-distributive. So $p = p \odot q^\alpha \leq q^\alpha$. Thus, q^α is the largest idempotent (in the orders $\leq$ and $\preceq$) such that $q^\alpha \preceq q$. Then, by Proposition 2(6), $q^- = q^\alpha$. □

Now, we are able to define an upper lower approximation:

DEFINITION 7

Let Q be a commutative and semicartesian quantale that is also unital and 'double-distributive'. For each $q \in Q$, define:

$$q^+ := \bigwedge \{p \in Q : q \leq q \odot p\} = \bigwedge \{p \in Q : q \preceq p\}.$$

LEMMA 1
If Q is unital, semicartesian and double-distributive, then

1. $1^+ = 1, 0^+ = 0$
2. $q \preceq q^+$
3. $q \leq q^+$
4. $q \odot q^+ = q$
5. $q = q^+ \Leftrightarrow q \odot q = q$
6. $q^+ \odot q^+ = q^+$
7. $q^{++} = q^+$
8. q^+ is the $\preceq$-least $y \in Q$ such that $q \preceq y$ and is $\odot$-idempotent (i.e. $q = q \odot y$ and $y \odot y = y$).
9. $(a \odot b)^+ \leq a^+ \odot b^+$
10. $x \preceq y \Rightarrow x^+ \preceq y^+ \Leftrightarrow x^+ \leq y^+$
11. Let $\{q_i : i \in I\} \subseteq Q$, then
12. $q_j{}^+ \odot \bigvee_{i \in I} q_i \geq q_j$
13. $q_j \odot \bigvee_{i \in I} q_i^+ = q_j$
14. $(\bigvee_j q_j) \preceq \bigvee_i q_i{}^+$
15. $(\bigvee_i q_i)^+ \leq \bigvee_i q_i{}^+$
16. Suppose Q divisible, then
17. $x \leq y \Rightarrow x^+ \leq y^+$
18. $(\bigvee_i q_i)^+ = \bigvee_i q_i{}^+$

PROOF. The proof follows the same spirit of Proposition 2. We just check the two properties that require Q to be divisible:

15. Since Q is divisible, $a \leq b$ implies there exist $x \in Q$ such that $a = b \odot x$. Then $a^+ = (b \odot x)^+ \leq b^+ \odot x^+ \leq b^+$.

16. The non-trivial inequality depends on the hypothesis of Q divisible: $(\bigvee_i q_i)^+ \geq \bigvee_i q_i{}^+$. This follows from (15) since $q_j \leq \bigvee_i q_i$, then $(q_j)^+ \leq (\bigvee_i q_i)^+$ and then, by sup property $\bigvee_j (q_j^+) \leq (\bigvee_i q_i)^+$ □

3 Sheaves on quantales

We present sheaves on quantales as a functor that satisfies gluing properties, which are formally expressed by an equalizer diagram. This is similar to the definition of sheaves on idempotent quantales proposed in [2], but it is a completely different case since we are interested in semicartesian quantales: if the quantale was idempotent *and* semicartesian we would obtain a locale, by Proposition 1, and the theory of sheaves on locales is already well established.

From now on we always consider commutative semicartesian quantales, unless stated otherwise.

Remind that every quantale can be seen as a poset category Q where the objects are elements of Q and the morphism $v \to u$ is given by the order relation $v \leq u$.

DEFINITION 8
A **presheaf on a quantale** Q is a functor $F : Q^{op} \to Set$.

Given $u, v \in Q$ such that $v \leq u$, we consider restriction maps $\rho^u_v : F(u) \to F(v)$ and denote $\rho^u_v(s) = s_{\restriction_v}$, for any $s \in F(u)$. The functoriality of a presheaf F gives us:

1. $\forall u \in Q$ and $\forall s \in F(u)$, $s_{\restriction_u} = s$
2. $\forall w \leq v \leq u$ in Q and $\forall s \in F(u)$, $s_{\restriction_w} = (s_{\restriction_v})_{\restriction_w}$

We will freely manipulate the restriction maps using the above notation.

Before we present the definition of sheaves on quantales, we recall the correspondent definition for a locale $\mathcal{L}$.

Definition 9

A presheaf $F : \mathcal{L}^{op} \to Set$ is a **sheaf** if for any cover $u = \bigvee_{i \in I} u_i$, of any element $u \in \mathcal{L}$, the following diagram is an equalizer

$$F(u) \xrightarrow{e} \prod_{i \in I} F(u_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{(i,j) \in I \times I} F(u_i \wedge u_j),$$

where

$$e(t) = \{t{\restriction_{u_i}} \ : \ i \in I\}, \quad p((t_k)_{k \in I}) = (t_i{\restriction_{u_i \wedge u_j}})_{(i,j) \in I \times I}$$
$$q((t_k)_{k \in I}) = (t_j{\restriction_{u_i \wedge u_j}})_{(i,j) \in I \times I}.$$

Remark 6

The cover $u = \bigvee_{i \in I} u_i$ in $\mathcal{L}$ is a cover in the sense of a Grothendieck pretopology. If $\mathcal{L} = \mathcal{O}(X)$ is the locale of open sets of a topological space, it is immediate how to obtain the usual notion of a sheaf on a topological space X from the above definition. We will further explore this remark later.

Our approach to define a sheaf on a commutative and semicartesian quantale is simple: we replace the meet operation $\wedge$ by the multiplication $\odot$ of the quantale.

Definition 10

A presheaf $F : Q^{op} \to Set$ is a **sheaf** if for any cover $u = \bigvee_{i \in I} u_i$ of any element $u \in Q$, the following diagram is an equalizer:

$$F(u) \xrightarrow{e} \prod_{i \in I} F(u_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{(i,j) \in I \times I} F(u_i \odot u_j),$$

where

$$e(t) = \{t{\restriction_{u_i}} \ : \ i \in I\}, \quad p((t_k)_{k \in I}) = (t_i{\restriction_{u_i \odot u_j}})_{(i,j) \in I \times I}$$
$$q((t_k)_{k \in I}) = (t_j{\restriction_{u_i \odot u_j}})_{(i,j) \in I \times I}.$$

Remark 7

The cover $u = \bigvee_{i \in I} u_i$ in Q is not a cover in the sense of a Grothendieck pretopology. If $Q = \mathcal{I}(R)$ is the quantale of ideals of a commutative ring with unity R, we may mimic the sheaf theory of a topological space but for a ring.

Observe that the maps $F(u_i) \to F(u_i \odot u_j)$ exist because $u_i \odot u_j$ always is less or equal to u_i and u_j, for all $i, j \in I$. This is where we use the semicartesianity.

We write, respectively, $PSh(Q)$ and $Sh(Q)$ for the categories of presheaves and sheaves on Q, where the objects are, respectively, presheaves and sheaves, and the morphisms are natural transformations between them.

REMARK 8

The category of sheaves $Sh(Q)$ is a full subcategory of the category of presheaves $PSh(Q)$, i.e. the inclusion functor $i : Sh(Q) \to PSh(Q)$ is full.

Now we develop the firsts steps toward a sheaf theory on quantales, following the presentation of [7] in the case of locales.

Let F be a presheaf on Q.

DEFINITION 11

Let $(u_i)_{i\in I}$ be a family of elements of Q. We say a family $(s_i \in F(u_i))_{i\in I}$ of elements of F is **compatible** if for all $i,j \in I$ we have

$$s_i{\restriction_{u_i \odot u_j}} = s_j{\restriction_{u_i \odot u_j}}.$$

DEFINITION 12

We say a presheaf F is **separated** if, given $u = \bigvee_{i\in I} u_i$ in Q and $s, s' \in F(u)$, we have

$$\left(\forall i \in I\, s{\restriction_{u_i}} = s'{\restriction_{u_i}}\right) \implies (s = s').$$

Using compatible families we equivalently define:

DEFINITION 13

Let $u = \bigvee_{i\in I} u_i$ in Q and $(s_i \in F(u_i))_{i\in I}$ a compatible family in F, we say the presheaf F is a **sheaf** if exists a unique element $s \in F(u)$ (called the gluing of the family) such that $s{\restriction_{u_i}} = s_i$, for all $i \in I$.

It is a straightforward exercise in category theory to show that the definitions 10 and 13 are equivalent.

Next we provide a list of results that have exact the same proof as in the case of sheaves on locales. See Lemmas 2.1.5, 2.1.6, 2.1.7 and 2.1.8 in [7].

PROPOSITION 9

Let F be a presheaf on Q.

1. F is a sheaf iff F is a separated presheaf and given $u = \bigvee_{i\in I} u_i$ in Q, every compatible family $(s_i \in F(u_i))_{i\in I}$ can be glued into an element $s \in F(u)$ such that $s{\restriction_{u_i}} = s_i$, for all $i \in I$.
2. If F is separated, $F(0)$ has at most one element. If F is a sheaf, $F(0)$ has exactly one element.
3. If $u = \bigvee_{i\in I} u_i$ in Q and $s \in F(u)$, then the family $\left(s{\restriction_{u_i}}\right)_{i\in I}$ is compatible.
4. Let F be a sheaf on Q and $\{u_i \in Q \,:\, i \in I\}$ a partition of u. Then $F(u) \cong \prod_{i\in I} F(u_i)$.

The following constructions provide sheaves over a quantale Q from a sheaf over Q and any $u \in Q$.

PROPOSITION 10

Let F be a sheaf on a quantale Q and $u \in Q$. For each $w \leq v$, consider:

$$F{\restriction_u}(v) = \begin{cases} F(v), & \text{if } v \leq u \\ \emptyset, & \text{otherwise,} \end{cases}$$

$$F_{\restriction_u}(w \le v) = \begin{cases} F(w \le v) : F(v) \to F(w), & \text{if } w \le v \le u \\ ! : \emptyset \to F_{\restriction_u}(w), & \text{if } w \le v \not\le u, \end{cases}$$

is a sheaf.[3]

PROOF. It is clear that $F_{\restriction_u}$ is a presheaf. Consider $v = \bigvee_{i \in I} v_i$ in Q, and $s, s' \in F_{\restriction_u}(v)$ such that $s_{\restriction_{v_i}} = s'_{\restriction_{v_i}}, \forall i \in I$

If $v \le u$, then $s,' s \in F(v) = F_{\restriction_u}(v)$. Since F is a sheaf, it is separated so $s = s'$. If $v \not\le u$, then $s,' s \in \emptyset$ and there is nothing to do. Thus, $F_{\restriction_u}$ is a separated presheaf.

Now consider $(s_i \in F_{\restriction_u}(v_i))_{i \in I}$ a compatible family. Suppose $F_{\restriction_u}(v_i) = \emptyset$ for some $i \in I$. For such $i \in I$, there is no s_i in $F_{\restriction_u}(v_i)$, then, there is $j \in I$ such that $s_{i_{\restriction_{u_i \odot u_j}}} \neq s_{j_{\restriction_{u_i \odot u_j}}}$. In other words, the family is not compatible. This implies $F_{\restriction_u}(v_i) = F(v_i)$, for all $i \in I$. So $v_i \le u$, which means $\bigvee_{i \in I} v_i = v \le u$. Therefore, $F_{\restriction_u}(v) = F(v)$.

Since F is a sheaf, we conclude the compatible family $(s_i \in F_{\restriction_u}(v_i))_{i \in I}$ can be glued into $s \in F_{\restriction_u}(v_i)$ such that $s_{\restriction_{v_i}}, \forall i \in I$. By Proposition 9, $F_{\restriction_u}$ is a sheaf. □

PROPOSITION 11

Let F be a sheaf on a quantale Q and $u \in Q$. For each $w \le v$, the following presheaf[4] is a sheaf:

$$F^{(u)}(v) := F(u \odot v)$$

$$F^{(u)}(w \le v) := F(u \odot w \le u \odot w)$$

PROOF. It is clear that $F^{(u)}$ is a presheaf, since F is a sheaf and $w \le v$ in Q implies that $(u \odot w) \le (u \odot v)$.

Note that if $v = \bigvee_i v_i$ then $u \odot v = \bigvee_i (u \odot v_i)$ is a cover.

Take a family $(s_i) \in F^{(u)}(v_i) = F(u \odot v_i)$, such that

$$F(u \odot v_i \odot v_j \le u \odot v_i)(s_i) = F(u \odot v_i \odot v_j \le u \odot v_j)(s_j) \in F(u \odot v_i \odot v_j)\, \forall i \in I$$

Since $u \odot v_i \odot u \odot v_j \le u \odot v_i \odot v_j$, we have that $s_i \in F(v_i \odot u)$ is a compatible family for F. Since F is a sheaf, there is a unique gluing $s \in F(u \odot v) = F^{(u)}(v)$ for the family $(s_i)_{i \in I}$. □

EXAMPLE 6

The functor $Q(-, v)$ is a sheaf, for every fixed $v \in Q$.

Recall that $Q(-, v)$ is the functor $Hom_Q(-, v)$ so it is a presheaf, where if $w \le u$, then we send the unique element $\{(u \le v)\}$ in $Q(u, v)$ to the unique element $\{(w \le v)\}$ in $Q(w, v)$.

Observe that we have two cases:

1. Suppose $u \le v$: since $u_i \le u$, for all $i \in I$, we have that $u_i \le v$, for all $i \in I$. Take $s_i = (u_i \to v) \in Q(u_i, v)$, since $u_i \odot u_j \le u_i, u_j$, for all $i, j \in I$,

$$s_{i_{\restriction_{u_i \odot u_j}}} = (u_i \odot u_j \to v) = s_{j_{\restriction_{u_i \odot u_j}}}.$$

[3] Note that $F_{\restriction_1} = F$ and if $u' \le u \in Q$, then $F_{\restriction_u \restriction_{u'}} = F_{\restriction_{u'}}$.

[4] Note that $F^{(1)} = F$ and if $u', u \in Q$, then $(F^{(u)})^{(u')} = F^{(u' \odot u)}$.

So $(s_i)_{i\in I}$ is a compatible family. To conclude $Q(u,v)$ is a sheaf, take the only element $s = (u \to v) \in Q(u,v)$ and observe that $s_{\restriction_{u_i}} = (u_i \to v) = s_i$, for all $i \in I$.

2. Suppose $u \not\leq v$: if $u_i \leq v$, for all $i \in I$, by definition of supremum, $\bigvee_{i\in I} u_i \leq v$, which is not possible. So there is at least one $i \in I$ (if $I \neq \emptyset$) such that $u_i \not\leq v$. Thus, $Q(u,v)$ and $Q(u_i,v)$ are empty sets, for such an $i \in I$. Then the sheaf condition is vacuously true.
3. If $I = \emptyset$, then $\bigvee\limits_{i\in\emptyset} u_i = 0$ and $Q(0,v)$ fits in the first case since $0 \leq v$.

PROPOSITION 12

Construction of sheaves.

1. Let $(Q_j)_{j\in J}$ be a family of commutative and semicartesian quantales and $(F_j)_{j\in J}$ be a family of sheaves, $F_j : Q_j^{op} \to Set$, for each $j \in J$. Then: $\prod_{j\in J} Q_j$ is a commutative semicartesian quantal; a family $\{(u_j^i)_{j\in J} : i \in I\}$ is a cover of $(u_j)_{j\in J} \in \prod_{j\in J} Q_j$ iff for each $j \in J$, $\{u_j^i : i \in I\}$ is a cover of $u_j \in Q_j$; and $\prod_{j\in J} F_j : (\prod_{j\in J} Q_j)^{op} \to Set$ given by $(\prod_{j\in J} F_j)(u_j)_{j\in J} := \prod_{j\in J} F_j(u_j)$ is a sheaf with the restriction maps defined component-wise from each F_j.
2. Let $F : Q^{op} \to Set$ be a sheaf on the commutative and semicartesian quantale Q. Let $e, a \in Q$, $e \leq a$, $e^2 = e$ and consider $Q' = [e,a]$, the (commutative and semicartesian) 'subquantal' of Q. Then $F' : Q'^{op} \to Set$ defined by $F'(u) = F(u)$, if $u \neq e$ and $F'(e) = \{*\}$, with non-trivial restriction maps $F'(v) \to F'(u) = F(v) \to F(u)$, if $e < v \leq u$, is a sheaf.

PROOF.

1. Straightforward.
2. Let $u = \bigvee_{i\in I} u_i$ be a cover in Q', and $(s_i \in F'(u_i))_{i\in I}$ a compatible family. Since $F'(u_i) = F(u_i)$ and the restriction maps for F' are restriction maps for F, we have that $(s_i \in F(u_i))_{i\in I}$ a compatible family. Since F is a sheaf, there is a unique gluing $s \in F(u) = F'(u)$. So F' is a sheaf. □

Next we introduce a concrete example of a sheaf.

EXAMPLE 7

Take $Q = ([0,\infty], +, \geq)$ the extended half-line quantale. Let (X,d) be an (extended) metric space. For each $A \subseteq X$ and each $r \in [0,\infty]$ consider balls $F_A(r) = B_r(A) = \{x \in X : d(x,A) \leq r\}$. Note that $s \geq r$ entails $B_r(A) \subseteq B_s(A)$ and, in the obvious way $F_A : [0,\infty] \to Set$ became a presheaf over the quantale Q where $F_A((s \geq r)) : F_A(r) \hookrightarrow F_A(s)$ is the inclusion. Moreover, this is a sheaf, since if $r = \bigwedge_{i\in I} s_i$ in $[0,\infty]$, then the diagram below is an equalizer

$$B_r(A) \to \prod_i B_{s_i}(A) \rightrightarrows \prod_{i,j} B_{s_i+s_j}(A)$$

for non-empty coverings. However, if $I = \emptyset$, then $r = \bigwedge_{i\in I} s_i = \infty$. Therefore, $B_\infty(A)$ is not single element (i.e. is not the terminal object in *Set*). This means that the sheaf condition fails when $I = \emptyset$. To surpass this, we maintain our definition $B_r(A)$ for all $r \in [0,\infty)$ but for $r = \infty$ we define $B_\infty(A) = \{*\}$. For any $s \geq r$, the restrictions map is the identity map on $\{*\}$.

Now we prove categorical properties of $Sh(Q)$ that are classic results in the localic case. We start with a list of statements whose verification follows exact the same steps of the localic case. See [7, Chapter 2] for details.

PROPOSITION 13
We have that:

1. The subcategory $Sh(Q) \hookrightarrow Set^{Q^{op}}$ is closed under limits;
2. $Sh(Q)$ has a terminal object, the (essentially unique) presheaf such $card(\mathbf{1}(u)) = 1$, for each $u \in Q$. Moreover, if Q is unital, then $Q(-,1) \cong \mathbf{1}$;
3. A monomorphism between sheaves $\eta : F \rightarrowtail G$ is just a monomorphism between their underlying presheaves (and they are monomorphism if and only if $\eta_u : F(u) \to G(u)$ is injective, for each $u \in Q$);
4. Every morphism $\eta : Q(-,v) \to F$, where F is a (pre)sheaf is, automatically, a monomorphism;
5. The family of representable sheaves $Q(-,u)$, indexed by elements of Q, is a set of generators for $Sh(Q)$.

PROPOSITION 14
We have that:

1. For each $v, v' \in Q$, there is at most one (mono)morphism $Q(-,v) \to Q(-,v')$ and this exists precisely when $v \leq v'$.
2. If H is a sheaf and $\epsilon : H \rightarrowtail Q(-,v)$ is a monomorphism, then $H \cong Q(-,h)$ where $h = \bigvee\{u \leq v : H(u) \neq \emptyset\}$.

PROOF.

1. For each $u, v \in Q$, note that $card(Q(u,v)) \in \{0,1\}$.
   Suppose there is a morphism $\eta : Q(-,v) \to Q(-,v')$. So, for all $u \in Q$ we have $\eta_u : Q(u,v) \to Q(u,v')$. If $Q(u,v') = \emptyset$, then $Q(u,v) = \emptyset$. Thus, if $u \leq v$, then $u \leq v'$. In particular, for $u = v$, we obtain $v \leq v'$.
   Conversely, if $v \leq v'$, consider $i_{v,v'} : Q(-,v) \to Q(-,v')$. For all $u \in Q$, we have $i_{v,v'}(u) : Q(u,v) \to Q(u,v')$.
   If $u \not\leq v$, then $Q(u,v) = \emptyset$ and $i_{v,v'}(u) : \emptyset \to Q(u,v')$ the unique function from the $\emptyset$, since the $\emptyset$ is an initial object in *Set*.
   If $u \leq v$, since $v \leq v'$, $u \leq v'$ and then $i_{v,v'}(u)(u \leq v) = (u \leq v')$. For any other morphism $j_{v,v'} : Q(-,v) \to Q(-,v')$, we obtain that $j_{v,v'}(u) : \emptyset \to Q(u,v')$ the unique function from the $\emptyset$, whenever $u \not\leq v$ and $j_{v,v'}(u)(u \leq v) = (u \leq v')$, whenever $u \leq v$. So $i_{v,v'} = j_{v,v'}$.
2. Since ϵ is a monomophism, ϵ_u is injective and then $card(H(u)) \in \{0,1\}$ for each $u \in Q$ with $H(u) = \emptyset$ whenever $u \not\leq v$. So let
   $$h = \bigvee\{u \leq v : H(u) \neq \emptyset\} = \bigvee\{u \in Q : H(u) \neq \emptyset\}.$$
   We will show that $H(u)$ is non-empty only when $u \leq v$. Note that:
   - If $q \leq p$ and $H(p) \neq \emptyset$, then $H(q) \neq \emptyset$ (since H is a presheaf);
   - Since $card(H(h)) = 1$, we have $H(h) \neq \emptyset$. Once $H(p), H(q) \neq \emptyset$ entails $H(p \odot q) \neq \emptyset$, by the sheaf condition we have an equalizer diagram between two parallel arrows where the source and target are both singletons.
   Therefore, $H(u) \neq \emptyset$ iff $u \leq h$. Now, we will show that $H(u) \to Q(u,h)$ is a (unique) bijection, for each $u \in Q$.
   If $u \not\leq h$, then $\emptyset = H(u) \to Q(u,h) = \emptyset$. If $u \leq h$, then $H(u)$ and $Q(u,h)$ are both singletons. So ϵ_u is an injection and a surjection in *Set*, therefore, a bijection for all $u \in Q$ and then ϵ is an isomorphism.

□

4 Sh(Q) is not a topos

We remind the reader that a Grothendieck topos is any category equivalent to the category $Sh(\mathcal{C}, J)$ of sheaves on a category $\mathcal{C}$ with a Grothendieck (pre)topology J. The pair $(\mathcal{C}, J)$ is called a site. In particular, $Sh(L)$ is a Grothendieck topos where $\mathcal{C} = L$ is the poset category by the locale L and we define the Grothendieck pretopology by $\{f_i : U_i \to U\} \in J(U) \iff u = \bigvee_{i\in I} u_i$. However, if $u = \bigvee_{i\in I} u_i$ and $v \leq U$ for u, u_i, v in a semicartesian quantale, we may have $v \neq u \odot v = \bigvee_{i\in I} u_i \odot v$. This means that the *stability axiom* in the definition of a Grothendieck pretolopogy is not satisfied in the quantalic case. Nevertheless, there could exist another site to provide that $Sh(Q)$ is a Grothendieck topos.

In this section, we study deeper categorial properties that make $Sh(Q)$ even more similar to a Grothendieck topos, but we also show that it is not a Grothendieck topos. The argument relies on the fact that every Grothendieck topos is an *elementary topos*—cartesian closed category with pullbacks, a terminal object and a subobject classifier. It is well known that if A is an object of a topos $\mathcal{E}$, then $Sub_{\mathcal{E}}(A)$ is a Heyting Algebra[5] [7, Proposition 6.2.1]. We will prove that $Sub_{Sh(Q)}(Q(-,a)) \cong [0,a]$ is an isomorphism of quantales. Since $[0,a]$ is not a Heyting Algebra in general when a is not an idempotent element, then $Sh(Q)$ is not a topos.

The first property we want for $Sh(Q)$ is that it has a set $(G_i)_{i\in I}$ of strong generators. Remind that:

Definition 14

[6, Definition 5.2.1] A category $\mathcal{M}$ is locally λ-presentable, for a regular cardinal λ, when

1. $\mathcal{M}$ is cocomplete;
2. $\mathcal{M}$ has a set $(G_i)_{i\in I}$ of strong generators;
3. each generator G_i is λ-presentable.

It is known that any Grothendieck topos is a λ-locally presentable category for some regular cardinal λ [7, Proposition 3.4.16]. We were able to show that $Sh(Q)$ also is λ-locally presentable, where $\lambda = max\{card(Q)^+, \aleph_0\}$, but we will let a detailed proof about this for a future paper. We do sketch the argument for the reader used with the terminology: use y to denote the Yoneda embedding and consider a covering $\{u_i : i \in I\}$. The Day convolution gives a monoidal structure in $PSh(Q)$ such that we have 'projections' $y(u_j) \otimes y(u_k) \xrightarrow{p_1^{jk}} y(u_j)$ and $y(u_j) \otimes y(u_k) \xrightarrow{p_2^{jk}} y(u_k)$. Since $u_i \leq u$, for each $i \in I$,there is an arrow $y(u_i) \xrightarrow{\phi_i} y(u)$. Denote by $y(u_j) \underset{y(u)}{\otimes} y(u_k)$ the equalizer of $\phi_j \circ p_1^{jk}$ and $\phi_k \circ p_2^{jk}$. Thus, we define a sieve $S(\{u_i\})$ of a covering $\{u_i : i \in I\}$ as the following coequalizer in $PSh(Q)$:

$$\coprod_{j,k} y(u_j) \underset{y(u)}{\otimes} y(u_k) \rightrightarrows \coprod_i y(u_i) \longrightarrow S(\{u_i\})$$

with the coproduct on the left being taken over $y(u_j) \underset{y(u)}{\otimes} y(u_k) \rightarrowtail y(u_j) \otimes y(u_k) \cong y(u_i \odot u_j)$. Then we orthogonalize the class of morphism $\{S(\{u_i\}) \to y(u) : u = \bigvee_{i\in I} u_i\}$ and observe that such orthogonalization corresponds to the category of sheaves on Q. This shows that $Sh(Q)$ is a λ-orthogonality class in $PSh(Q)$. By the Theorem 1.39 of [1], we conclude

[5] The class of complete Heyting algebras and of locales coincide.

Corollary 1
$Sh(Q)$ is:

1. a reflective subcategory of $PSh(Q)$ closed under λ-directed colimits,
2. locally λ-presentable

The first item provides that the inclusion functor $i : Sh(Q) \to PSh(Q)$ has a left adjoint functor $a : PSh(Q) \to Sh(Q)$, which we will call **sheafification functor**. The second item provides that $Sh(Q)$ has a set of strong generators.

First, we use the sheafification to show that $Sh(Q)$ admits a monoidal (closed) structure. We will use the notions of *normal reflective embedding* and *normal enrichment for a reflective embedding*. The definitions are available at [9], but we reproduce they here with a different notation:

Definition 15
Let $\psi \dashv \phi : D \to B$ be an adjoint pair.

1. $\psi \dashv \phi$ is a reflective embedding if ϕ is full and faithful on morphism.
2. When B has a fixed monoidal closed structure the reflective embedding is called **normal** if there exists a monoidal closed structure on D and monoidal functor structures on ψ and ϕ for which ϕ is a normal closed functor and the unit and counit of the adjunction are monoidal natural transformations.

Given that B is a category with a fixed monoidal structure, by saying that a reflective embedding $\psi \dashv \phi : D \to B$ *admits a normal enrichment* we mean that there are conditions for $\psi \dashv \phi$ be normal. In particular, the functor ψ carries the monoidal structure from B to C in a compatible and unique (up to monoidal isomorphism) way, see [9] for details. Also, the reader may find the definition of a normal closed functor in [4] or be satisfied by statement that the inclusion functor is a normal closed functor. Then we state Proposition 1.1 in [9], with a different notation.

Proposition 15
Let $C = (C, \otimes, I)$ be a small monoidal category. A reflective embedding $\psi \dashv \phi : D \to PSh(C)$ admits normal enrichment if and only if the functor $F(U \otimes -)$ is isomorphic to some object in D whenever F is a object of D and U is an object of C.

Proposition 16
The sheafification admits normal enrichment.

Proof. In this case, $C = Q$ is the posetal category of quantales. By Proposition 11, the functor $F(u \odot -)$ is a sheaf for every $u \in Q$, whenever F is a sheaf. By Proposition 15, the reflective embedding $a \dashv i$ admits a normal enrichment. ☐

Monoidal structure in $Sh(Q)$**:** The above result gives that $Sh(Q)$ has a monoidal closed structure where $F \otimes G := a(i(F) \otimes_{Day} i(G))$, for F, G sheaves on Q.

Now, we state two results from Borceux to obtain more information about $Sh(Q)$:

Proposition 17
[5, Proposition 4.5.15] If $\mathcal{C}$ has finite limits and possesses a strong set of generators, so $\mathcal{C}$ is well-powered (i.e. for all C object of $\mathcal{C}$, the subobjects $Sub(C)$ of $\mathcal{C}$ forms a set).

Proposition 18
[5, Corollary 4.2.5] In a complete and well-powered category, $Sub(C)$ has all infima/intersections and suprema/unions.

COROLLARY 2
$Sh(Q)$ is a complete and well-powered category, and for all F sheaf on Q, $Sub(F)$ has all infima/intersections and suprema/unions.

COROLLARY 3
Factorization of morphisms in $Sh(Q)$:

For each morphism $\phi : F \to G$ in $Sh(Q)$, there exists the least subobject of G, represented by $\iota : G' \rightarrowtail G$, such that $\phi = \iota \circ \phi'$ for some (and thus, unique) morphism $\phi' : F \to G'$. Moreover, ϕ' is an *epimorphism*.

PROOF. By the previous results, there exists the extremal factorization above $\phi = \iota \circ \phi'$, such that $\iota : G' \rightarrowtail G$ is a mono. To show that $\phi' : F \to G'$ is an epi, consider $\eta, \epsilon : G' \to H$ such that $\eta \circ \phi' = \epsilon \circ \phi'$ and let $\gamma : H' \rightarrowtail G'$ be the equalizer of η, ϵ. Then, by the universal property of γ, there exists a unique $\phi'' : F \to H'$ such that $\gamma \circ \phi'' = \phi'$. On the other hand, by the extremality of ι, there exists a unique $\gamma' : G' \to H'$ such that $\iota = \iota \circ \gamma \circ \gamma'$. Since ι is a mono, we obtain that $\gamma \circ \gamma' = id_{G'}$. Thus, γ is a mono that is a retraction: this means that $\gamma = eq(\eta, \epsilon)$ is an iso, i.e. $\epsilon = \eta$. Thus ϕ' is an epi. □

REMARK 9
Keeping the notation above, if $\phi : F \to G$ is already a mono then, by the extremality of $\iota : G' \rightarrowtail G$, $\phi \cong \iota$ and thus $\phi' : F \to G'$ is an isomorphism. It is natural to ask ourselves if the converse holds in general. Conversely, does it hold that any morphism that is mono and epi is an iso? This would mean that the category $Sh(Q)$ is *balanced*.

Any category with factorizations (extremal mono, epi) and where all the monos are regular (i.e. monos are equalizers) is balanced. A 'topos-theoretic' way to show that all monos are regular is to show that there exists a 'universal mono' $true : 1 \rightarrowtail \Omega$ that is a subobjects classifier. We will address this question in the next section.

THEOREM 1
Assume that Q is unital. We have the following isomorphisms of complete lattices:

$$h_Q \colon Q \to Represented(Sh(Q))$$
$$q \mapsto Q(-,q)$$

$$i_Q \colon Represented(Sh(Q)) \to Representable(Sh(Q))/isos$$
$$Q(-,q) \mapsto [Q(-,q)]_{iso}$$

$$j_Q \colon Representable(Sh(Q))/isos \to Sub(\mathbf{1})$$
$$[R]_{iso} \mapsto [R \cong Q(-,q) \rightarrowtail Q(-,1) \cong \mathbf{1}]_{iso}.$$

Thus, $k_Q = j_Q \circ i_Q \circ h_Q : Q \to Sub(\mathbf{1})$ is an isomorphism of complete lattices.

More generally, take any $a \in Q$, we may amend the map k_Q in a way that it sends $b \in [0,a]$ to $[Q(-,b) \rightarrowtail Q(-,a)]_{iso}$, then we obtain a quantalic isomorphism $k_a : [0,a] \to Sub(Q(-,a))$.

PROOF. We will just show that h_Q, i_Q, j_Q are isomorphisms of posets, and, since Q is a complete lattice, then h_Q, i_Q, j_Q are complete lattices isomorphisms.

h_Q is isomorphism: By the very definition of represented functor, the map h_Q is surjective. For injectivity see that $Q(-,q) = Q(-,p)$ implies that $Q(u,q) = Q(u,p)$, for all $u \in Q$, and so $p = q$.

Yoneda's lemma and Proposition 14.1 establishes that it preserves and reflects order since $p \leq q$ iff there is some (unique) (mono)morphism $\eta : Q(-,p) \to Q(-,q)$.

i_Q is isomorphism: Since it is a quotient map, it is surjective. i_Q is injective: by Proposition 14.2, $Q(-,p) \cong Q(-,q)$ implies $p = q$ and thus $Q(-,p) = Q(-,q)$. The map preserves and reflects order: this is a direct consequence of Proposition 14.1.

j_Q is isomorphism: Since $! : Q(-,1) \to \mathbf{1}$ is an isomorphism, we will just prove that j'_Q: $Representable(Sh(Q))/isos \to Sub(Q(-,1))$ $[R]_{iso} \mapsto [R \cong Q(-,q) \rightarrowtail Q(-,1)]_{iso}$ is an isomorphism. By the very definition of $Sub(F) = Mono(F)/isos$, it is clearly injective. Take $\eta : R \rightarrowtail Q(-,1)$, by Proposition 14.1, $R \cong Q(-,q)$, thus j'_Q is surjective. Now let R and R' be representable functors, there is a morphism $\eta : R \to R'$ iff this morphism is unique and it is a monomorphism, thus j'_Q preserves and reflects order. □

Definition 16

For each F sheaf on Q, we define the following binary operation on $Sub(F)$: Given $\phi_i : F_i \rightarrowtail F$, $i = 0, 1$ define $\phi_0 * \phi_1 : F_0 * F_1 \rightarrowtail F$ as the mono in the extremal factorization of $F_0 \otimes_F F_1 \rightarrowtail F_0 \otimes F_1 \rightrightarrows F$.

Theorem 2

For each $a \in Q$, the poset $Sub(Q(-,a))$, endowed with the binary operation $*$ defined above is a commutative and semicartesian quantale. Moreover, the poset isomorphism $k_a : [0,a] \to Sub(Q(-,a)), q \mapsto [Q(-,q)]_{iso}$, established in Theorem 1, is a quantale isomorphism.

Proof. As a consequence of the proof of Theorem 1, this map is well-defined, bijective and preserves and reflects orders; thus, it is a complete lattice isomorphism. It remains to show that $Q(-,u \odot v) \cong Q(-,u) * Q(-,v)$, for all $u, v \leq a$. We have that $Q(-,u) * Q(-,v) \rightarrowtail Q(-,a)$ is the mono in the extremal factorization of the arrow

$$Q(-,u) \otimes_{Q(-,a)} Q(-,v) \xrightarrow{equ} Q(-,u) \otimes Q(-,v) \rightrightarrows Q(-,a).$$

By Day convolution,

$$Q(-,u) \otimes Q(-,v) \cong Q(-,u \odot v).$$

Since $u \odot v \leq a$, by Proposition 14, there is unique (mono)morphism $Q(-,u \odot v) \to Q(-,a)$. So $Q(-,u) \otimes Q(-,v) \rightrightarrows Q(-,a)$ corresponds to $Q(-,u \odot v) \rightrightarrows Q(-,a)$. Thus, the parallel arrows coincide and then

$$Q(-,u) \otimes_{Q(-,a)} Q(-,v) \cong Q(-,u) \otimes Q(-,v) \cong Q(-,u \odot v).$$

Hence, the arrow

$$Q(-,u) \otimes_{Q(-,a)} Q(-,v) \to Q(-,a)$$

is isomorphic with the unique mono

$$Q(-,u \odot v) \rightarrowtail Q(-,a).$$

This shows that $Q(-,u \odot v) \cong Q(-,u) * Q(-,v)$, as we wish. □

This has an interesting direct application:

COROLLARY 4
Let Q be the quantale of ideals of a ring R, then Q is isomorphic to $Sub(Q(-,R))$.

So we can recover any ideal of R by analysing the subobjects of the sheaf $Q(-,R)$.

Summarizing this section: Take any commutative, semicartesian and unital quantale Q that is not a locale (= complete Heyting algebra)—there are plenty of such quantales—and consider the sheaf $\mathbf{1} \cong Q(-,1)$. By the Theorems 1 and 2, $Sub(Q(-,1)) \cong Sub(\mathbf{1})$ is isomorphic to Q as a quantale. Since Q is not a locale, then $Sub(\mathbf{1})$ is not a Heyting algebra. Therefore, from a well-known result in topos theory, we can conclude that $Sh(Q)$ is not even an elementary topos.

5 On the subobject classifier

The subobject classifier in a category may be seen as it *internal truth values object*. By definition, every elementary topos has a subobject classifier and we use it to construct the internal logic of a topos. In the category on sheaves on a locale L, the subobject classifier is the sheaf $\Omega(u) = \{q \in L : q \leq u\}$ such that for all $v \leq u$, we map q to $q \wedge v$, and $\top : \mathbf{1} \to \Omega$ defined by $\top_u(*) := u$ is the 'universal subobject'. Thus, for every F sheaf on L, we have a natural isomorphism $\eta_F : Sub(F) \to Sh(L)(F,\Omega)$ that sends equivalence class of monos $m : S \to F$ to its unique characteristic map $\chi_m : F \to \Omega$ (the proof of [16, Proposition 1, Chapter I.3] verifies that η_F is a natural iso). In particular, $Sh(L)(\mathbf{1},\Omega) \cong Sub(\mathbf{1}) \cong L$ explains how Ω, which is the 'internal truth values object' of $Sh(L)$, encodes the 'external truth values' of $Sh(L)$, the locale $L \cong Sub(\mathbf{1})$.

For general semicartesian and commutative quantales, just replacing the infimum by the quantalic multiplication does not yields a sheaf. Here we present two constructions—Ω^- and Ω^+—that provide approaches of a subobject classifier in $Sh(Q)$, in different subclasses of commutative semicartesian quantales, and pointing advantages and drawbacks of each one.

PROPOSITION 19
Let Q be a commutative, semicartesian and *geometric* quantale. For each $u \in Q$ define $\Omega^-(u) = \{q \in Q : q \odot u^- = q\}$ then, with the restriction map

$$\begin{aligned} \Omega^-(u) &\to \Omega^-(v) \\ q &\mapsto q \odot v^- \end{aligned}$$

for all $v \leq u$ in Q, Ω is a sheaf.

PROOF. Note that $q \odot v^- \in \Omega^-(v)$ since $q \odot v^- \odot v^- = q \odot v^-$. It is a presheaf because $q \odot u^- = q$ and, given $w \leq v \leq u$, $q \odot v^- \odot w^- = q \odot w^-$. The separability also holds: suppose $u = \bigvee_{i \in I} u_i$ and take $p, q \in \Omega^-(u)$ such that $p_{\restriction_{u_i}} = q_{\restriction_{u_i}}$ for all $i \in I$. Then

$$\begin{aligned} p = p \odot u^- &= p \odot (\bigvee_{i\in I} u_i)^- = p \odot \bigvee_{i\in I} u_i^- = \bigvee_{i\in I} p \odot u_i^- \\ &= \bigvee_{i\in I} q \odot u_i^- = q \odot \bigvee_{i\in I} u_i^- = q \odot (\bigvee_{i\in I} u_i)^- = q \odot u^- \\ &= q. \end{aligned}$$

The gluing is $q = \bigvee_{i \in I} q_i$, where $q_i \in \Omega^-(u_i)$ for each $i \in I$. Observe that $q \in \Omega^-(u)$:

$$q \odot u^- = \bigvee_{i \in I} q_i \odot (\bigvee_{j \in I} u_j)^- = \bigvee_{i \in I} q_i \odot \bigvee_{j \in I} u_j^- = \bigvee_{i \in I} q_i \odot u_i^- \odot \bigvee_{j \in I} u_j^- = \bigvee_{i \in I} q_i = q,$$

where we used that the quantale is geometric in the second equality, the fact $q_i \in \Omega^-(u_i)$ in the third and the idempotence of u_i^- in the forth.

Now we check that q is the gluing. On the one hand,

$$q_j = q_j \odot u_j^- \leq q \odot u_j^- = q_{\restriction_{u_j}}.$$

On the other hand, recording that $(u \odot v)^- = (u^- \odot v^-)$ by Proposition 2.10,

$$\begin{aligned} q_{\restriction_{u_j}} &= q \odot u_j^- = (\bigvee_{i \in I} q_i) \odot u_j^- = \bigvee_{i \in I} (q_i \odot u_j^-) = \bigvee_{i \in I} (q_i \odot u_i^- \odot u_j^-) \\ &= \bigvee_{i \in I} (q_i \odot (u_i \odot u_j)^-) = \bigvee_{i \in I} q_{i \restriction_{u_i \odot u_j}} = \bigvee_{i \in I} q_{j \restriction_{u_i \odot u_j}} \\ &= \bigvee_{i \in I} (q_j \odot (u_i \odot u_j)^-) = \bigvee_{i \in I} (q_j \odot u_i^- \odot u_j^-) = (\bigvee_{i \in I} u_i^-) \odot q_j \odot u_j^- \\ &= (\bigvee_{i \in I} u_i)^- \odot q_j = u^- \odot q_j \leq q_j. \end{aligned}$$

□

REMARK 10

1. The mapping $Q \mapsto \Omega^-$ preserves products and interval constructions (see Proposition 12).
2. Note that for each $v, u \in Q$, such that $v^- = u^-$, then $\Omega^-(v) = \Omega^-(u)$. In particular, if $u^- \leq v \leq u$, then $\Omega^-(v) = \Omega^-(u)$ and, moreover, $\Omega^-(u^-, v) = \Omega^-(v, u) = id_{\Omega^-(v)}$.
3. For each $u \in Q$, let $\bot_u, \top_u : 1(u) \to \Omega^-(u)$, where $\bot_u(*) := 0 \in \Omega^-(u)$ and $\top_u(*) := u^- \in \Omega^-(u)$. Then $\bot, \top : 1 \to \Omega^-$ are natural transformations.
4. For each $u \in Q$ and $v \in \Omega^-(u)$, we have $v^- \in \Omega^-(u)$: this defines a map $-_u : \Omega^-(u) \to \Omega^-(u)$, Then $()^- := (-_u)_{u \in Q} : \Omega^- \to \Omega^-$ is a natural transformation and $\top^- := ()^- \circ \top = \top$, $\bot^- := ()^- \circ \bot = \bot$.
5. If Q is a locale, then $\Omega^-(u) = \{q \in Q : q \odot u^- = q\} = \{q \in Q : q \leq u\} = \Omega_0(u)$, and $\top_u(*) = u^- = u$. Thus, $\top : 1 \to \Omega^-$ coincides with the subobject classifier in the category of sheaves on locales [7, Theorem 2.3.2]. We will readdress this subject bellow.

Our investigations did not lead to Ω^- being a subobjects classifier, but it does classifies the dense subobjects:

DEFINITION 17
A morphism of sheaves $\eta : G \to F$ is **dense** whenever $\forall u \in Q \forall y \in F(u) \exists m \in Q, u^- \leq m \leq u$ such that $F(m \leq u)(y) \in range(\eta_m)$ iff $y \in range(\eta_u)$

Note that, since $m \leq u, y \in range(\eta_u) \implies F(m \leq u)(y) \in range(\eta_m)$.

It can be easily verified that a sufficient condition to a morphism of sheaves $\eta : G \overset{\cong}{\to} F$ be a dense is: $\forall u \in Q \exists m \in Q, u^- \leq m \leq u$ such that the diagram below is a pullback:

$$\begin{array}{ccc} G(u) & \xrightarrow{\eta_u} & F(u) \\ {\scriptstyle G(m\leq u)}\downarrow & & \downarrow{\scriptstyle F(m\leq u)} \\ G(m) & \xrightarrow[\eta_m]{} & F(m) \end{array}$$

EXAMPLE 8
(Dense morphisms)

1. Every isomorphism $\eta : G \overset{\cong}{\to} F$ is a dense (mono)morphism.
2. If a point $\pi : 1 \to F$ is such that $\forall u \in Q \exists m \in Q, u^- \leq m \leq u, F(m \leq u) : F(u) \to F(m)$ is bijective, then $\pi : 1 \to F$ is a dense monomorphism. In particular, every point $\pi : 1 \to \Omega^-$ is a dense monomorphism.
3. Let $a, b \in Q$. If $b \leq a$, let $\eta : Q(-, b) \to Q(-, a)$ be the unique monomorphism (an inclusion, in fact). Then η is a dense monomorphism iff $\forall u \in Q \forall y \in [u, a] \exists m \in Q, u^- \leq m \leq u, (y \in [m, b] \Leftrightarrow y \in [u, b])$; therefore, taking $m = u$, we have that $Q(-, b) \hookrightarrow Q(-, a)$ is a dense inclusion.

We register the following (straightforward) result:

PROPOSITION 20
A pullback of a dense (mono)morphism in $Sh(Q)$ is a dense (mono)morphism.

THEOREM 3
Suppose that Q is a (commutative, semicartesian and) geometric quantale. Then the sheaf Ω^- introduced in Proposition 19 essentially classifies the dense subobject in the category $Sh(Q)$. More precisely:

1. $\top : 1 \to \Omega^-$, given by $\top_u : \{*\} \to \Omega^-(u), \top_u(*) = u^-$ determines a dense monomorphism in $Sh(Q)$.
2. For each dense monomorphism of sheaves $m : S \rightarrowtail F$, there is a unique morphism $\chi^m : F \to \Omega^-$, such that $\chi_m^- = \chi_m$, and such the diagram below is a pullback. Moreover, for each morphisms $\phi, \phi' : F \to \Omega^-$ that determine pullback diagrams, it holds: $\phi^- = \phi'^-$.

PROOF.

1. Since u^- is an idempotent, then $u^- \in \Omega^-(u)$ (in fact, $u^- = max\Omega^-(u)$). If $v \leq u$ then, by Proposition 2.9) $v^- \odot u^- = v^-$, thus $\top = (\top_u)_{u \in Q}$ is a natural transformation. By Proposition 13.4, it is clear that $\top$ is a monomorphism of sheaves. In Example 8 item 2, we argued that $\top$ is dense.
2. First note that since $\top$ is a dense monomorphism, it follows from Proposition 20 that the pullback of a morphism $\phi : F \to \Omega^-$ through $\top$ must be a *dense* monomorphism $m : S \rightarrowtail F$. Now, note that it is enough establish the result for dense subsheaves $i_S : S \hookrightarrow F$. We will split the proof in two parts, but first we will provide some relevant definitions and calculations.
   For each $u \in Q$ and $y \in F(u)$, define:
   $\langle y, u \rangle := \{v \in \Omega^-(u) : F(v \leq u)(y) \in S(v)\}$;
   $u_y := \bigvee \langle y, u \rangle$.

(a) If $v, w \in \Omega^-(u)$ and $w \leq v$, then $v \in \langle y, u\rangle \Rightarrow w \in \langle y, u\rangle$: S is a subpresheaf of F. In particular, if $v \in \langle y, u\rangle$, then $v^- \in \langle y, u\rangle$, since $(\,)^- : \Omega^- \to \Omega^-$ natural transformation, and $v^- \leq v$.
(b) If $\{v_i : i \in I\} \subseteq \langle y, u\rangle$, then $\bigvee_i v_i \in \langle y, u\rangle$: since $\Omega^-(u)$ is closed under suprema and S is a subsheaf of F.
(c) $u_y \in \langle y, u\rangle$ (by (b)) and $u_y^- \in \langle y, u\rangle$ (by (a)). Thus:
$u_y = max\langle y, u\rangle$ and $u_y^- = max(\langle y, u\rangle \cap Idem(Q))$.
(d) $u_y^- = \bigvee\{e \in Idem(Q) : \exists v, v \preceq u^-, e = v^-, F(v \leq u)(y) \in S(v)\}$, since Q is a geometric quantale, we have $u_y^- = (\bigvee\{v \in \Omega^-(u) : F(v \leq u)(y) \in S(v)\})^- = \bigvee\{v^- : v \in \Omega^-(u), F(v \leq u)(y) \in S(v)\}$.
Candidate and uniqueness: Suppose that $\phi : F \to \Omega^-$ is a natural transformation such that the diagram below is a pullback (where S is dense subsheaf of F).

$$S \overset{i_S}{\hookrightarrow} F \overset{\phi}{\to} \Omega^- \overset{\top}{\leftarrow} 1 \overset{!_S}{\leftarrow} S$$

Note that if $u^- \leq m \leq u$ then, by naturality, $\phi_m(F(m \leq u)(y)) = \phi_u(y) \odot m^- = \phi_u(y) \odot u^- = \phi_u(y)$.
Claim (i): It holds that $u_y^- \leq \phi_u(y) \leq u_y$. Moreover, if $\phi_u(y) \in Idem(Q)$, then $\phi_u(y) = u_y^-$.
Since the diagram is a pullback and limits in $Sh(Q)$ are pointwise (see Proposition 13.1), then for each $w \in Q$:

$$x \in S(w) \Leftrightarrow x \in F(w) \textit{ and } \phi_w(x) = w^-.$$

Thus, if $v \leq u$ is such that $F(v \leq u)(y) \in S(v)$, then by naturality:

$$v^- = \phi_v(F(v \leq u)(y)) = \phi_u(y) \odot v^-.$$

Note that $u_y \leq u^- \leq u$ and $u_y \in \langle y, u\rangle$, thus $u_y^- = \phi_u(y) \odot u_y^-$ and $u_y^- \leq \phi_u(y)$.
By naturality: $\phi_{\phi_u(y)}(F(\phi_u(y), u)(y))) = \phi_u(y) \odot \phi_u(y)^- = \phi_u(y)^-$
$\phi_u(y) \in \langle y, u\rangle$: since $\phi_u(y) \in \Omega^-(u)$ and $\phi_{\phi_u(y)}(F(\phi_u(y), u)(y))) = \phi_u(y)^-$ then, by the pullback condition, we have that $F(\phi_u(y), u)(y)) \in S(\phi_u(y))$, thus $\phi_u(y) \in \langle y, u\rangle$.
$\phi_u(y) \leq u_y$: since $\phi_u(u) \in \langle y, u\rangle$ and $u_y = max\langle y, u\rangle$.
$\phi_u(y) = u_y^-$, whenever $\phi_u(y) \in Idem(Q)$: this holds because we have established above that $\phi_u(y) \in \langle y, u\rangle$, $u_y^- \leq \phi_u(y) \leq u_y$ and because $u_y^- = max(\langle y, u\rangle \cap Idem(Q))$.
Thus, if $\phi_u(y) \in Idem(Q)$, then $\phi_{\phi_u(y)}(F(\phi_u(y), u)(y))) = \phi_u(y) = u_y^-$.
Claim (ii): If $\phi : F \to \Omega^-$ determines a pullback diagram, then $\phi^- = (\,)^- \circ \phi$ still determines a pullback. This holds because $x \in S(w)$ iff ($x \in F(w)$ and $\phi_w(x) = w^- = (w^-)^- = \phi_w^-(x)$).
Combining Claim (ii) and Claim (i), $\phi_u^-(y) = u_y^-$ for each $u \in Q$ and $y \in F(u)$, establishing the required uniqueness assertions.
Existence: For each $u \in Q$ and $y \in F(u)$, define $\chi_u^S(y) := u_y^-$. Then $(\chi_u^S)_{u \in Q}$ is a natural transformation and it determines a pullback diagram.
First, we will verify that $(\chi_u^S)_{u \in Q}$ is a natural transformation. Let $u, v \in Q$ be such that $v \leq u$ and let $y \in F(u)$. We have to show that:

$$\chi_v^S(F(v \leq u)(y)) = \chi_u^S(y) \odot v^-.$$

This means:

$$max(Idem(Q) \cap \langle F(v \leq u)(y), v\rangle) = v^- \odot max(Idem(Q) \cap \langle y, u\rangle).$$

On the one hand, note that $v^- \odot u_y^- = v^- \odot max(Idem(Q) \cap \langle y, u\rangle) = v^- \odot \bigvee(Idem(Q) \cap \langle y, u\rangle) = \bigvee\{v^- \odot e : e^2 = e = e \odot u^-, F(e \leq u)(y) \in S(e)\}$. Denoting $e' := v^- \odot u_y^-$, we have $e'^2 = e' = e' \odot v^-$ and $F(e' \leq v)(F(v \leq u)(y)) \in S(e')$, thus $e' = v^- \odot u_y^- \in Idem(Q) \cap \langle F(v \leq u)(y), v\rangle$ and then

$$max(Idem(Q) \cap \langle F(v \leq u)(y), v\rangle) \geq v^- \odot max(Idem(Q) \cap \langle y, u\rangle).$$

On the another hand, denote $e'' = max(Idem(Q) \cap \langle F(v \leq u)(y), v\rangle)$. Then, $e''^2 = e'' = e'' \odot v^-$ and $F(e'' \leq v)(F(v \leq u)(y)) \in S(e'')$. Then, $e'' \in Idem(Q), e'' \leq v^- \leq u^-$ and $e'' \in \langle y, u\rangle$. Thus, $e'' \in Idem(Q)$ and $e'' \leq v^-, u_y^-$. Then, $e'' = e'' \odot e'' \leq v^- \odot u_y^-$, i.e. $max(Idem(Q) \cap \langle F(v \leq u)(y), v\rangle) \leq v^- \odot max(Idem(Q) \cap \langle y, u\rangle)$.

Now we show that the pullback condition holds for each $u \in Q$:

$$y \in S(u) \;\Leftrightarrow\; (y \in F(u) \text{ and } u^- = \chi_u^S(y) = u_y^-).$$

On one hand, let $y \in S(u)$, then $y \in F(u)$ and $u^- \in \Omega^-(u)$ is such that $F(u^- \leq u)(y) \in S(u^-)$, since S is a subpresheaf of F. Then $u^- \in Idem(Q) \cap \langle y, u\rangle$. Thus, by (b), $u^- \leq u_y^-$. On the other hand $u_y^- \in \Omega^-(u)$, thus $u_y^- \leq u^-$. Summing up: $\chi_u^S(y) = u_y^- = u^-$.

On the another hand, let $y \in F(u)$ be such that $u^- = \chi_u^S(y) = u_y^-$. Then $u^- = max(\langle y, u\rangle \cap Idem(Q))$. Therefore, $F(u^- \leq u)(y) \in S(u^-)$ and, since $i_S : S \hookrightarrow F$ is a dense inclusion, we have $y \in S(u)$. □

Using $(-)^+$ instead of $(-)^-$, we actually obtain a subobject classifier, $\top : \mathbf{1} \to \Omega^+$, but this requires extra conditions on Q. Thus, similar to the localic case, there is a pair of inverse bijections $Sh(Q)(F, \Omega^+) \rightleftarrows Sub(F)$, that are natural in F. In particular, $Sh(Q)(\mathbf{1}, \Omega^+) \cong Sub(\mathbf{1}) \cong Q$ explains how the 'internal truth values object' of $Sh(Q)$, Ω^+, encodes the 'external truth values' of $Sh(Q)$, the quantale $Q \cong Sub(\mathbf{1})$.

We construct it as follows.

Let Q be a unital, commutative, semicartesian and double distributive quantale.

(a) For each $u \in Q$, define

$$\Omega^+(u) := \{q \in Q : q^+ \odot u = q\} = \{q \in Q : q^+ \odot u \leq q \text{ and} q \leq u\}.$$

(b) Note that:

- $0 = min(\Omega^+(u)), u = max(\Omega^+(u))$.
- $\Omega^+(u) = \{q \in Q : q^+ \odot u \leq q \text{ and} q \leq u\}$.

If $q = q^+ \odot u$, then $q \leq u$ and $q^+ \odot u \leq q$. If $q \leq u$ and $q^+ \odot u \leq q$, then $q = q^+ \odot q \leq q^+ \odot u \leq q$.

- $\Omega^+(u) = \{q' \in Q : \exists e, e \odot e = e, q' = e \odot u\}$.

If $q \in \Omega^+(u)$, then $q = q^+ \odot u$ (by definition) and $q^+ \in Idem(Q)$. If $e \odot e = e$, then $e \odot u \leq u$ and $(e \odot u)^+ \odot u \leq (e^+ \odot u^+) \odot u = e \odot (u^+ \odot u) = e \odot u$. Then, by the previous item, $e \odot u \in \Omega^+(u)$.

- $\Omega^+(u) \subseteq \{q \in Q : q \odot u^+ = q \text{ and } q \leq u\}$. (†)

If $q \in \Omega^+(u)$, then $q = q^+ \odot u = q^+ \odot u \odot u^+ = q \odot u^+$.

- If $q \in \Omega^+(u)$, then $q^+ \odot u^+ = q^+$ and $q^+ \in \Omega^+(u^+)$.

Suppose that $q \in \Omega^+(u)$. By the previous observation $q = q \odot u^+$ and, by the minimality of q^+, we have $q^+ \leq u^+$. Since q^+, u^+ are idempotents, we obtain $q^+ = q^+ \odot u^+ = q^{++} \odot u^+$, thus $q^+ \in \Omega^+(u^+)$.

(c) Given $v \leq u$ in Q, we can define a corresponding restriction map by:

$$\begin{aligned} \Omega^+(u) &\to \Omega^+(v) \\ q &\mapsto q^+ \odot v \end{aligned}$$

Note that:

- $q^+ \odot v \in \Omega^+(v)$. Because, in general, for every $e \in Idem(Q)$, $e \odot v \in \Omega^+(v)$.
- If $v \leq u$ and $q \in \Omega^+(u)$, then $q^+ \odot v \in \{q' \in Q : q' \odot u^+ = q' \text{ and } q' \leq u\} \supseteq \Omega^+(u)$. Because $q^+ \odot v \leq q^+ \odot u = q \leq u$ and $q^+ \odot v = q^+ \odot v \odot v^+ \leq q^+ \odot v \odot u^+ \leq q^+ \odot v$.
- If $v^+ \odot u = v$, then $v \leq u$ and $\Omega^+(v) \subseteq \Omega^+(u)$. Because $v = v^+ \odot u \leq u$ and if $e \in Idem(Q)$, then $e \odot v = e \odot (v^+ \odot u) = (e \odot v^+) \odot u$ and the result follows from a characterization given in (b).

(d) Ω^+ is a presheaf on Q.

The definition of $\Omega^+(u)$ gives $q = q^+ \odot u$, $\forall u \in Q$, $\forall q \in \Omega^+(u)$. In other words, $q_{\restriction_u} = q$.

Concerning the composition, if $w \leq v \leq u$, on the one hand,

$$\begin{aligned} q_{\restriction_w} &= q^+ \odot w \\ &= q^+ \odot w \odot w^+ && w \odot w^+ = w \\ &\leq q^+ \odot v \odot w^+ && w \leq v \\ &\leq (q^+ \odot v)^+ \odot w^+ && (-)^+ \text{ is a supremum} \\ &= (q_{\restriction_v})_{\restriction_w}. \end{aligned}$$

On the other hand,

$$\begin{aligned} (q_{\restriction_v})_{\restriction_w} &= (q^+ \odot v)^+ \odot w \\ &\leq q^{+^+} \odot v^+ \odot w && (a \odot b)^+ \leq a^+ \odot b^+ \\ &= q^+ \odot v^+ \odot w \\ &\leq q^+ \odot w \\ &= q_{\restriction_w}. \end{aligned}$$

(e) Ω^+ is a separated presheaf.

Let $u = \bigvee_{i \in I} u_i$, and $p, q \in \Omega^+(u)$ such that $q_{\restriction_{u_i}} = p_{\restriction_{u_i}}$, $\forall i \in I$. Then:

$$\begin{aligned} q = q^+ \odot u = q^+ \odot \bigvee_{i \in I} u_i = \bigvee_{i \in I} q^+ \odot u_i = \bigvee_{i \in I} q_{\restriction_{u_i}} \\ = \bigvee_{i \in I} p_{\restriction_{u_i}} = \bigvee_{i \in I} p^+ \odot u_i = p^+ \odot \bigvee_{i \in I} u_i = p^+ \odot u = p. \end{aligned}$$

Based on the construction of the (separated) presheaf Ω^+ provided above, we obtain the following result.

PROPOSITION 21
Let Q be a unital, commutative, semicartesian and double distributive quantale. If we assume, in addition, that Q is a divisible quantale satisfying the coherence property below:

$$(coherence) \qquad \forall a, b, a', b' \in Q,\ a \leq b, a' \leq b',$$

$$a \odot b' = a' \odot b \Rightarrow a^+ \odot b'^+ = a'^+ \odot b^+.$$

Then Ω^+ is a sheaf.

PROOF. We have established in the construction above that Ω^+ is a separated presheaf. Thus, by Proposition 9.1, to conclude that Ω^+ is a sheaf, it is enough to show that every compatible family for Ω^+ admits some gluing.

Note that if $q \in \Omega^+(u)$ and $v \leq u$, then

$$q^+ \odot v \in \{q' \in Q : q' \odot u^+ = q' \text{ and } q' \leq u\} \supseteq \Omega^+(u).$$

It follows from $q^+ \odot v \leq q^+ \odot u = q \leq u$, from Q being divisible, and that $v \leq u$ implies $(q^+ \odot v) \odot u^+ = q^+ \odot (v \odot u^+) = q^+ \odot (v \odot v^+) = q^+ \odot v$, by Lemma 1.14.

Now we will verify that any compatible family can be glued.

Let $u = \bigvee_{i \in I} u_i$ and consider $\{q_i \in \Omega^+(u_i)\}_{i \in I}$ be a compatible family.

This means $q_i^+ \odot (u_i \odot u_j) = q_j^+ \odot (u_i \odot u_j)$ and this is equivalent to

$$(compatibility)\ q_i \odot u_j = q_j \odot u_i,\ \forall i, j \in I.$$

Set $q := \bigvee_{i \in I} q_i^+ \odot u$.

We have that $q \in \Omega^+(u)$ because, by item (b):

$$q = \bigvee_{i \in I} q_i^+ \odot u \leq u$$

$$q^+ \odot u = (\bigvee_{i \in I} q_i^+ \odot u)^+ \odot u \leq \bigvee_{i \in I} (q_i^+ \odot u)^+ \odot u \leq \bigvee_{i \in I} (q_i^+ \odot u^+) \odot u) = \bigvee_{i \in I} q_i^+ \odot u = q.$$

We have $q_j \leq q_{\restriction u_j}$. Indeed:
$q_j = q_j^+ \odot u_j = (q_j^+ \odot u_j)^+ \odot u_j \leq (\bigvee_{i,k \in I} q_i^+ \odot u_k)^+ \odot u_j = (\bigvee_{i \in I} q_i^+ \odot u)^+ \odot u_j = q_{\restriction u_j}$.

It remains to prove that $q_{\restriction u_j} \leq q_j$. We will use the extra hypothesis on the quantale Q—that it satisfies also the coherence property—to obtain this.

$$\begin{aligned}
q_{\restriction u_j} &= q^+ \odot u_j \\
&= (\bigvee_{i \in I} q_i^+ \odot u)^+ \odot u_j && \text{by definition of } q \\
&= (\bigvee_{i \in I} (q_i^+ \odot u)^+) \odot u_j && \text{by Lemma 1.16} \\
&\leq (\bigvee_{i \in I} (q_i^{++} \odot u^+)) \odot u_j && \text{by Lemma 1.9} \\
&= (\bigvee_{i \in I} (q_i^+ \odot u^+)) \odot u_j && \text{by Lemma 1.7} \\
&= \bigvee_{i \in I} q_i^+ \odot (u^+ \odot u_j) && \text{by distributivity and associativity} \\
&= \bigvee_{i \in I} q_i^+ \odot u_j && \text{by Lemmas 1.12 } \textit{and } \text{1.15} \\
&= \bigvee_{i \in I} (q_i^+ \odot u_j^+) \odot u_j && \text{by Lemma 1.4} \\
&= \bigvee_{i \in I} (q_j^+ \odot u_i^+) \odot u_j && \textit{by (compatibility) and (coherence)} \\
&= (\bigvee_{i \in I} u_i^+) \odot (q_j^+ \odot u_j) \\
&= (u^+) \odot (q_j) && \text{by Lemma 1.16} \\
&= q_j
\end{aligned}$$

where the last equality holds because $q_j \geq (u^+) \odot (q_j) \geq (u_j^+) \odot (q_j) = q_j$ by Lemma 1.15 and item (b) (†) previously constructed.

Therefore, gluings exist, and this ends the proof. □

REMARK 11

1. There are some examples of quantales that satisfy all the conditions in the above construction: the locales, the quantales of ideals of any PID, the quantales $\mathbb{R}_+ \cup \{\infty\}$, $\mathbb{N} \cup \{\infty\}$, etc. Moreover, note that this class of quantales is closed under arbitrary products and that the mapping $Q \mapsto \Omega^+$ preserves the product construction (see Proposition 12).
2. In the localic case, the sheaf Ω^+ coincides with the subobject classifier sheaf, which we denote by Ω_0. Note that if Q is a locale, then $\Omega^+(u) = \{q \in Q : q^+ \odot u = q\} = \{q \in Q : q \leq u\} = \Omega_0(u)$, and $\top_u(*) = u$. Thus, $\top : 1 \to \Omega^+$ coincides with the subobject classifier in the category of sheaves on locales [7, Theorem 2.3.2]. Moreover, note that, in general, the restriction $\Omega^+_{\restriction Idem(Q)} : Idem(Q)^{op} \to Set$ is such that for each $e \in Idem(Q)$ $\Omega^+_{\restriction Idem(Q)}(e) \cap$

$Idem(Q) = \{e' \in Idem(Q) : e' \leq e\} = \Omega_0(e)$ is the value of the subobject classifier of the localic topos $Sh(Idem(Q))$. We will readdress this subject in Theorem 4.

3. If Q is the quantale that satisfies the property: for each $q \in Q$, if $q > 0$, then $q^+ = \top$. Then $\Omega^+(u) = \{0, u\}$, for each $u \in Q$. Note that this condition holds whenever Q is the quantale of ideals of some PID or one of the linear quantales $\mathbb{R}_+ \cup \{\infty\}$, $\mathbb{N} \cup \{\infty\}$, for instance.

REMARK 12

1. Let Q be a commutative, semicartesian, unital and double distributive quantal. For each $v, u \in Q$ such that $v \leq u$ are equivalent: (i) $v \preceq (u \to v)$; (ii) $v^+ \leq (u \to v)$; (iii) $v^+ \odot u \leq v$; (iv) $v^+ \odot u = v$; (v) $v \in \Omega^+(u)$; (vi) $\Omega^+(v) \subseteq \Omega^+(u)$. See Remark 3 to remind the meaning of $u \to v$.
2. If a commutative quantale Q satisfies, for each $v \leq u$, the condition (i) above, then it will be called *strongly divisible*. Note that a quantale satisfying the hypothesis in previous item and that is strongly distributive, then (by item (iv)) it is divisible.
3. The class of quantales that are commutative, semicartesian, unital, double distributive and strongly divisible—and its subclass of quantales satisfying also the condition of *coherence* (21) – contains all locales and is closed under arbitrary products and under interval construction of the type $[e, 1]$, where $e \in Idem(Q)$.

THEOREM 4

Suppose that Q is a (commutative, semicartesian), unital, double-distributive, coherent and *strongly divisible* quantale.

Then the sheaf Ω^+ (see Proposition 21) classifies all the subobjects in the category $Sh(Q)$. More precisely:

1. $\top : 1 \to \Omega^+$, given by $\top_u : \{*\} \to \Omega^+(u), \top_u(*) = u$ determines a monomorphism in $Sh(Q)$.
2. For each monomorphism of sheaves $m : S \rightarrowtail F$, there is a *unique* morphism $\chi^m : F \to \Omega^+$, such and such the diagram below is a pullback.

$$S \overset{m}{\rightarrowtail} F \overset{\chi^m}{\to} \Omega^+ \overset{\top}{\leftarrow} 1 \overset{!_S}{\leftarrow} S$$

In particular, every monomorphism in $Sh(Q)$ is regular and the category $Sh(Q)$ is balanced (see Remark 9).

PROOF. The strategy of this proof is similar to the corresponding proof of Theorem 3, so we will just will provide details in some parts.

Let $i_S : S \hookrightarrow F$ be a subsheaf.

For each $u \in Q$ and each $y \in F(u)$ we define: $\langle u, y \rangle := \{q \in \Omega^+(u) : F(q \leq u)(y) \in S(q)\}$ and $u^y := \bigvee \langle u, y \rangle$. Then $u^y = max\langle u, y \rangle$, since the set $\langle u, y \rangle$ is closed under suprema (because $\Omega^+(u)$ is closed under suprema and S is a subsheaf of F).

Candidate and uniqueness:

Suppose that $\phi : F \to \Omega^+$ is a natural transformation such that the diagram

$$S \overset{i_S}{\hookrightarrow} F \overset{\phi}{\to} \Omega^+ \overset{\top}{\leftarrow} 1 \overset{!_S}{\leftarrow} S$$

is a pullback.

By naturality, note that for each $u, v \in Q$, if $v \leq u$ and $y \in F(u)$, then

$$\phi_v(F(v \leq u)(y)) = \phi_u(y)^+ \odot v \leq \phi_u(y)^+ \odot u = \phi_u(y).$$

By the pullback condition, if $v \leq u$ is such that $F(v \leq u)(y) \in S(v)$, then:

$$(*) \qquad v = \phi_v(F(v \leq u)(y)) = \phi_u(y)^+ \odot v.$$

Moreover, if $v \in \Omega^+(u)$, then:

$$(**) \qquad v \leq \phi_u(y) \odot v^+.$$

Indeed, $v = u \odot v^+ = u \odot (\phi_u(y)^+ \odot v)^+ \leq u \odot ((\phi_u(y)^+)^+ \odot (v)^+ = u \odot (\phi_u(y)^+) \odot v^+ = \phi_u(y) \odot v^+$.

Since $u^y \in \langle u, y \rangle$, we obtain, by $(*)$, $(\phi_u(y))^+ \odot u^y = u^y$ and, in particular, $u^y \leq \phi_u(y)^+$.

On the other hand, denote $\phi_u(y) = u' \in \Omega^+(u)$, then $u' = \phi_u(y) = \phi_{u'}(F(u' \leq u)(y))$, thus $u' \in \langle y, u \rangle$. Therefore:

$$(***) \qquad \phi_u(y) \leq u^y.$$

By the relation just above and taking $v = u^y$ in $(**)$, we obtain
$u^y \leq \phi_u(y) \odot u^{y+} \leq u^y \odot u^{y+} = u^y$, i.e. $u^y \in \Omega^+(\phi_u(y))$.
Thus:

$$(****) \qquad (\phi_u(y))^+ \odot u^y = u^y = \phi_u(y) \odot u^{y+}.$$

Summing up: $\phi_u(y) \leq u^y \leq (\phi_u(y))^+$. Therefore, $(\phi_u(y))^+ = (u^y)^+$.

Now we prove that the hypothesis of *strongly divisible* on Q entails $\phi_u(y) = u^y$: this establishes the uniqueness.

By $(***)$ and since $\phi_u(y) \leq u^y$ entails $\Omega^+(\phi_u(y)) \subseteq \Omega^+(u^y)$ and $\phi_u(y) \in \Omega^+(\phi_u(y))$, then $\phi_u(y) = \phi_u(y)^+ \odot u^y$. Thus by $(****)$ $\phi_u(y) = \phi_u(y)^+ \odot u^y = u^y$.

Existence:

For each $u \in Q$ and $y \in F(u)$, define

$$\chi_u^S(y) := u^y = max\{q \in \Omega^+(u) : F(q \leq u)(y) \in S(q)\} \in \Omega^+(u).$$

Since $u = max\Omega^+(u)$, then $y \in S(u)$ iff $\chi_u^S(y) = u$. Thus for each $u \in Q$, the map $\chi_u^S : F(u) \to \Omega^+(u)$ defines a pullback diagram in the category *Set*.

It remains only to check that $(\chi_u^S)_{u \in Q}$ is a natural transformation. Let $v \leq u$.

We always have $\chi_u^S(y)^+ \odot v \leq \chi_v^S(F(v \leq u)(y))$, since: $\chi_v^S(F(v \leq u)(y)) = max\{q' \in \Omega^+(v) : F(q' \leq v)(F(v \leq u)(y)) \in S(q')\}$ and $\chi_u^S(y)^+ \odot v \in \{q' \in \Omega^+(v) : F(q' \leq v)(F(v \leq u)(y)) \in S(q')\}$.

Indeed, $\chi_u^S(y) = u^y \in \{q \in \Omega^+(u) : F(q \leq u)(y) \in S(q)\} \subseteq \Omega^+(u)$. Thus, (i) $(u^y)^+ \odot v \in \Omega^+(v)$; (ii) $(u^y)^+ \odot v \leq (u^y)^+ \odot u = u^y$, thus $F((u^y)^+ \odot v \leq u)(y) \in S((u^y)^+ \odot v)$. Therefore, $\chi_u^S(y)^+ \odot v \in \Omega^+(v)$ and $F((u^y)^+ \odot v \leq v)(F(v \leq u)(y)) \in S((u^y)^+ \odot v)$.

Now, we will use the hypothesis that Q is strongly divisible to obtain $\chi_u^S(y)^+ \odot v \geq \chi_v^S(F(v \leq u)(y))$ and that establishing the naturality of $(\chi_u^S)_{u \in Q}$.

Let $v' = \chi_v^S(F(v \leq u)(y))$. Since $F(v' \leq u)(y) \in S(v')$ and $v' \in \Omega^+(v)$, by strongly divisibility of Q, $\Omega^+(v) \subseteq \Omega^+(u)$. Thus, $v' \in \langle y, u \rangle$ and, therefore, $v' \leq u^y$. Then, $\chi_v^S(F(v \leq u)(y)) \leq \chi_u^S(y)$.

Therefore, $\chi_v^S(F(v \leq u)(y)) = \chi_v^S(F(v \leq u)(y))^+ \odot v \leq \chi_u^S(y)^+ \odot v$, as we wish. □

6 Final remarks and future works

Note that for each (finite or infinite) cardinal θ, and for each sheaf F, there are 'external operations': $\bigvee_F^\theta, \bigwedge_F^\theta : Sub(F)^\theta \to Sub(F)$ and $*_F : Sub(F) \times Sub(F) \to Sub(F)$; moreover, these operation are natural in F. Thus, for the subclass of commutative semicartesian quantales in the conditions of Theorem 4, an application of Yoneda lemma gives us 'internal operations' (= morphism in $Sh(Q)$) $\bigvee^\theta, \bigwedge^\theta : (\Omega^+)^\theta \to \Omega^+$ and $* : \Omega^+ \times \Omega^+ \to \Omega^+$. Based on the relationship between such internal operations, we have started a study of the internal logic of the category $Sh(Q)$: these results will be part of a future work.

Also, we are developing a parallel work regarding change of basis for sheaves on quantales: suppose $\phi : Q \to Q'$ preserves $\odot$, suprema and unity (thus ϕ is a functor). Then we obtain a 'change of basis' functor $\phi_* : Sh(Q') \to Sh(Q)$

$$(F' \stackrel{\eta'}{\to} G') \mapsto (F' \circ \phi^{op} \stackrel{\eta'_\phi}{\to} G' \circ \phi^{op}).$$

Since limits in categories of sheaves are coordinatewise, this functor ϕ_* clearly preserves limits, thus it is natural to pose the question if it is a right adjoint. The candidate to be the left adjoint is the same one that appears in the localic case and so we want to study under what conditions the left Kan extension $Ran_{\phi^{op}} : PSh(Q) \to PSh(Q')$ restricts to $Ran_{\phi^{op}} : Sh(Q) \to Sh(Q')$. The technical construction will appear somewhere else but we leave here two motivations to go deeper into this question: (i) Kan extensions are known as the 'best approximation' of a given functor through another given functor. Since the inclusion $Idem(Q) \to Q$ preserves multiplication, suprema and unity we may argue that $Sh(Idem(Q))$ is the best localic topos associated to $Sh(Q)$ and (ii) study sheaves on rings by studying sheaves on topological spaces and vice-versa through functors between the locale of open subsets of a topological space X and the quantale of ideals of a ring generated by the X, as the ring of continuous functions, for instance. We are investigating such relations and applying cohomological techniques aiming to find closer relationships between algebra and geometry.

Moreover, we are exploring a notion of covering appropriate for monoidal categories that is more general than a Grothendieck pretopology such that the respective category of sheaves—called *Grothendieck lopos*—encompass both Grothendieck topos and $Sh(Q)$. This approach leads to the development of an *elementary lopos theory*, a generalization of toposes but with a linear internal logic. In [3], the coauthors of the present work describe a category of Q-Sets, which will also be used to guide the future definition of a lopos, and when Q is locale we obtain a well-known equivalence between the category of sheaves on a locale and complete Q-Sets. Therefore, this paper is one of many steps in the direction of a broader project.

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